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2D Boundary Element Method for Creep Problems Using Isoparametric Quadratic Elements

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Abstract

A two dimensional boundary element method (BEM) formulation based on an initial strain approach has been successfully implemented for creep problems using Norton-Bailey creep law. The creep problems of a square plate and a plate with a circular hole are investigated for primary and secondary creep using isoparametric quadratic elements to model the boundary with 3-node boundary elements, and to model the interior domain with 8-node quadrilateral cells. The results of problems above are compared with the finite element solutions using MSC.Marc software and the analytical solutions where available and shown to be in good agreement.

1. Introduction

The boundary element method (BEM) has been widely used to analyse both elastic and time-dependent inelastic engineering problems. Telles and Brebbia [1] presented the BEM formulation based on an initial strain approach for 2-D elastoplastic problems. Linear interpolation functions were employed for the boundary elements and the internal triangular cells. The von Mises yield criterion and the Prandtl-Reuss flow rule were applied for the plastic strain increment. The problems of a perforated aluminium strip under uniaxial tension, a polystyrene crazing problem under uniaxial and biaxial tension and plate strain punch were analysed. The results were compared and agreed well with the FEM and experimental results. Lee and Fenner [2] have presented the isoparametric quadratic boundary element formulation for two-dimensional elastoplastic analysis based on an initial strain approach. The problems of uniaxial tensile behaviour, bending behaviour, internally pressurised cylinder, perforated plate in tension, and uniaxial behaviour in cyclic plasticity were analysed. The results were compared to and agreed well with the analytical solutions, experimental results and the FEM. Banerjee and Raveendra [3] have proposed the boundary element formulation based on an initial stress approach for 2-D and 3-D elastoplastic problems. The quadratic isoparametric representation was used to model the boundary elements and the volume cells. The problems of 2-D and 3-D thick cylinder and 3-D thick sphere under internal pressure, 2-D and 3-D perforated strip under tension and 2-D notched plate under axial tension were analysed. The results agreed well with the FEM and experimental results. Telles and Brebbia [4] have presented the boundary element formulation based on an initial stress approach for 2-D (plane stress and plane strain) and 3-D viscoplasticity and creep problems. Euler's formula was used for time integration. The problems of a deep beam under uniform load, a thin disc under constant external edge load and a plate under thermal shrinkage were solved and compared with the FEM and the analytical solutions showing good agreement. Cathie and Banerjee [5] have presented the 3-D boundary element method for inelastic (plasticity and creep) problems. Two approaches, initial stress and initial strain, as well as the solution algorithm were introduced. A combined creep law which included both time hardening and strain hardening creep laws has been presented. The problems of square plates with and without holes under constant uniaxial and biaxial tension and a thick cylinder under internal pressure were analysed using a power law creep function. The boundary geometry and unknowns were represented by quadratic elements. The forward difference approximation (Euler) was implemented for time integration. The results agreed well with the exact solutions.

In this paper a BE formulation for creep and time-dependent material behaviour based on an initial strain approach is presented using Norton-Bailey creep laws. Isoparametric quadratic elements are used for the boundary element and domain cells.

2. 2D boundary element formulation for creep problems

2.1 Integral equation for creep problems

The BE formulation for creep is based on an initial strain approach which has the same form as that used for plasticity by replacing plastic strain rates by creep strain rates as follows (see, for example, Mukherjee [6]):

$$C_{ij}(P)\dot{u}_i(P) + \int_\Gamma T_{ij}(P,Q)\dot{u}_j(Q)d\Gamma(Q) = \int_\Gamma U_{ij}(P,Q)\dot{t}_j(Q)d\Gamma(Q) + \int_A W_{kij}(P,q)\dot{\varepsilon}^c_{ij}(q)dA(q) \tag{1}$$

where $\dot{u}_i$, $\dot{t}_i$ and $\dot{\varepsilon}^c_{ij}$ are the displacement, traction and creep strain rates, respectively. U_{ij}, T_{ij} and W_{kij} are the displacement , traction and third-order kernels, respectively, which are functions of the position of the load point P and the field point Q or the interior point q and the material properties. Γ and A are the boundary and surface of the solution domain. The algebraic expressions for the kernels U_{ij}, T_{ij} and W_{kij} can be found, for example, in Lee and Fenner [2].

2.2 Constitutive model.

The Norton-Bailey creep law for time hardening based on the Prandtl-Reuss flow rule is used and can be defined as follows (see, for example, Kraus [7] and Becker and Hyde [8]):

$$\dot{\varepsilon}^c_{ij} = \frac{3}{2} mB\left(\sigma_{eff}\right)^{(n-1)} S_{ij} t^{(m-1)} \tag{2}$$

where m, B and n are material constants dependent on temperature. σ_{eff} and S_{ij} are the effective stress and the deviatoric stress, respectively.

3. Numerical implementation.

To perform the integration in equation (1) numerically, the boundary and domain must be divided into a number of boundary elements and domain cells. It is convenient to use a new coordinate system that is local to the element using an intrinsic variable ξ with its origin at the midpoint node and values −1 and +1 at the end nodes. Figure 1 shows a typical three-node boundary element and a typical eight-node quadrilateral domain cell.

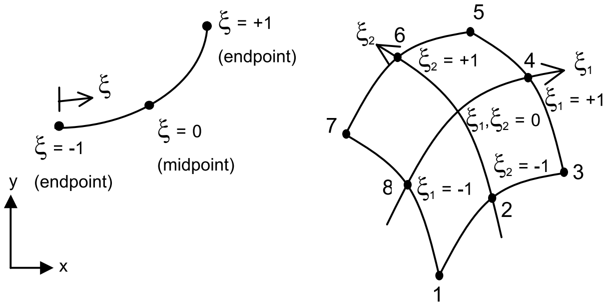

Figure 1 Isoparametric quadratic boundary element and cell.

Since isoparametric quadratic elements are used for the boundary and domain elements, the geometry and unknown variables have the same order and can be described using the appropriate shape functions. Therefore, the geometry and unknown variables on the boundary can be written as follows:

$$\begin{aligned} x_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(x_i)_c \\ \dot{u}_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(\dot{u}_i)_c \\ \dot{t}_i(\xi) &= \sum_{c=1}^{3} N_c(\xi)(\dot{t}_i)_c \end{aligned} \tag{3}$$

where $N_c(\xi)$ is the boundary quadratic shape function and is defined as follows:

$$\begin{aligned} N_1(\xi) &= \frac{-\xi}{2}(1-\xi) \\ N_2(\xi) &= (1+\xi)(1-\xi) \\ N_3(\xi) &= \frac{\xi}{2}(1+\xi) \end{aligned} \tag{4}$$

For the domain cells, two-dimensional quadratic shape functions are used as follows:

$$x_i(\xi_1,\xi_2)=\sum_{c=1}^{8}N_c(\xi_1,\xi_2)(x_i)_c$$

$$\dot{u}_i(\xi_1,\xi_2)=\sum_{c=1}^{8}N_c(\xi_1,\xi_2)(\dot{u}_i)_c \tag{5}$$

$$\dot{t}_i(\xi_1,\xi_2)=\sum_{c=1}^{8}N_c(\xi_1,\xi_2)(\dot{t}_i)_c$$

where the domain quadratic shape functions, $N_c(\xi_1,\xi_2)$, are defined as follows:

$$\begin{aligned}
N_1(\xi_1,\xi_2)&=\frac{-1}{4}(1-\xi_1)(1-\xi_2)(1+\xi_1+\xi_2)\\
N_2(\xi_1,\xi_2)&=\frac{1}{2}(1-\xi_1^2)(1-\xi_2)\\
N_3(\xi_1,\xi_2)&=\frac{-1}{4}(1+\xi_1)(1-\xi_2)(1-\xi_1+\xi_2)\\
N_4(\xi_1,\xi_2)&=\frac{1}{2}(1+\xi_1)(1-\xi_2^2)\\
N_5(\xi_1,\xi_2)&=\frac{-1}{4}(1+\xi_1)(1+\xi_2)(1-\xi_1-\xi_2)\\
N_6(\xi_1,\xi_2)&=\frac{1}{2}(1-\xi_1^2)(1+\xi_2)\\
N_7(\xi_1,\xi_2)&=\frac{-1}{4}(1-\xi_1)(1+\xi_2)(1+\xi_1-\xi_2)\\
N_8(\xi_1,\xi_2)&=\frac{1}{2}(1-\xi_1)(1-\xi_2^2)
\end{aligned} \tag{6}$$

The integral equation (1) can be discretised into boundary elements and domain cells, and written in terms of the local coordinates as follows:

$$\begin{aligned}
&C_{ij}(P)\dot{u}_i(P)+\sum_{m=1}^{M}\sum_{c=1}^{3}\dot{u}_j(Q)\int_{-1}^{+1}T_{ij}(P,Q)N_c(\xi)J(\xi)d\xi=\\
&\sum_{m=1}^{M}\sum_{c=1}^{3}\dot{t}_j(Q)\int_{-1}^{+1}U_{ij}(P,Q)N_c(\xi)J(\xi)d\xi+\\
&\sum_{d=1}^{D}\sum_{c=1}^{8}\dot{\varepsilon}_{ij}^{c}(q)\int_{-1}^{+1}\int_{-1}^{+1}W_{kij}(P,q)N_c(\xi_1,\xi_2)J(\xi_1,\xi_2)d\xi_1 d\xi_2
\end{aligned} \tag{7}$$

where M is the total number of the boundary elements and D is the total number of the domain cells. c is a node counter form 1 to 3 for boundary elements and 1 to 8 for domain cells. $J(\xi)$ and $J(\xi_1,\xi_2)$ are the Jacobians of transformation.

Taking each boundary node in turn as the load point P and performing the integrations, a set of linear algebraic equations can be written as follows:

$$[A][\dot{u}]=[B][\dot{t}]+[W][\dot{\varepsilon}^c] \tag{8}$$

where the matrices [A], [B] and [W] contain the integrals of the kernels T_{ij}, U_{ij}, and W_{kij}, respectively. For two-dimensional problems, if the total number of boundary nodes is N and the total number of the domain cell points is H, then the solution matrices [A] and [B] will be square matrices of size 2N x 2N, whereas the matrix [W] will be a rectangular matrix of size 2N x 3H. Unlike the FE method, all BE matrices are fully populated.

The parameter $C_{ij}(P)$ contributes only to the diagonal coefficients of the [A] matrix (i.e. when P is equal to Q). When the points P and Q do not coincide, the standard Gaussian quadrature can be used.

4. Convergence criterion

The Euler method is used to update the variables at each time step as follows:

$$y_{i+1} = y_i + \Delta t_i \dot{y}_i \tag{9}$$

Although it is relatively simple to implement, the Euler method is a very slow process if a constant time step is employed. To improve the convergence rate, an automatic time step control which will automatically select the next time step for the next calculation is implemented. The main idea is to compare the error, e, at each time step, with the two predefined errors or tolerances, the maximum error, e_{max}, and the minimum error, e_{min}, as follows:

(i) If $e > e_{max}$, the current time step is reduced by a factor of less than 1.0 and the analysis is repeated.

(ii) If $e_{max} > e > e_{min}$, the current time step is used for the next calculation.

(iii) If $e_{min} > e$, the current time step is increased by a factor of greater than 1.0 for the next calculation.

The creep strain error which occurs in each time step can be defined as follows (see, e.g. Mukherjee [16]):

$$e = \frac{\left|\Delta t_i(\dot{\varepsilon}_i^c - \dot{\varepsilon}_{i-1}^c)\right|}{\left|\varepsilon_i^c\right|} \tag{10}$$

where $\dot{\varepsilon}_i^c$ is the creep strain rate at *i* th step and ε_i^c is the total creep strain. Note that the stress rate can alternatively be used in equation (10) instead of the creep strain rate.

5. Boundary element creep algorithm

The BE algorithm for creep can be summarised in the following steps:

1. Solve the BE equations to obtain the elastic solution (equation (1) without the last term) and calculate the stresses and strains at all nodes (at the boundary and interior nodes).
2. Calculate the creep strain rates from equation (2) for time hardening. Note that, for the first calculation, time *t* is zero.
3. For a small time step, solve the BE equations for creep (equation (1)) and calculate the stress and strain rates at all nodes.
4. Check convergence. If the solution is not converged, reduce the current time step by a factor less than 1 and check the convergence again. This process will be repeated until convergence is achieved.
5. Update the variables using the Euler method.
6. Select the next time step. If the current error is between the minimum and maximum prescribed tolerances, the current time step is used for next calculation. If the current error is less than the minimum prescribed tolerance, the current time step is multiplied by a factor greater than 1 and is used for the next calculation.
7. Repeat steps 2-6 until the final time is reached.

More details of the creep algorithm can be found in Chandenduang [9].

6. Creep examples

All tests are performed for 1000 hours using the automatic time step control with the maximum and minimum stress tolerances of 10^{-1} and 10^{-2}, respectively. The initial time step of 10^{-3} hour and 6 integration points are used. The results are compared with analytical (Becker and Hyde [8]) and finite element (MSC.Marc [10]) solutions.

6.1 Square plate

Six cases involving a square plate under tension are tested. These tests include both primary creep and secondary creep. The dimensions of the square plate are 100 mm x 100 mm. The boundary and domain are divided into 4 boundary elements and 1 cell, respectively, as shown in Figure 2. The material properties and creep parameters are as follows:

Young's Modulus (E) = 200×10^3 MPa

Poisson's Ratio (ν) = 0.3

B = 3.125×10^{-16}

m = 1.0 for secondary creep

m = 0.5 for primary creep

n = 5

The boundary conditions are as follows:

$u_y = 0$ along line ab

$u_x = 0$ along line ad

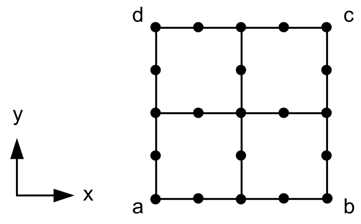

Figure 2 BE and FE mesh for the square plate (8 boundary elements and 4 cells).

Details of the 6 tests are listed below:

1. TEST1 and TEST2. The square plate is subjected to a uniaxial constant tensile stress of 200 N/mm^2 in the x-direction for TEST1 and to a uniaxial displacement of 0.1 mm in the x-direction for TEST2. Both tests are plane stress assumption and secondary creep law. The creep strains and stresses in the x-direction are plotted against time and shown in Figure 3 and Figure 4, respectively. The results show very good agreement with the error being less than 1%.

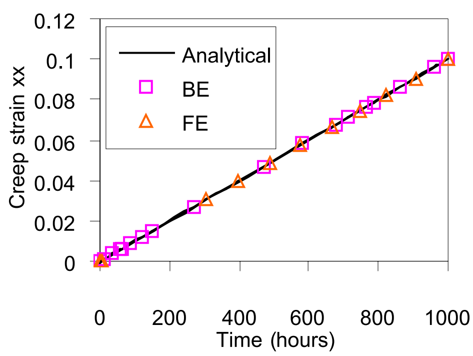

Figure 3 TEST1: creep strain, plane stress, secondary creep.

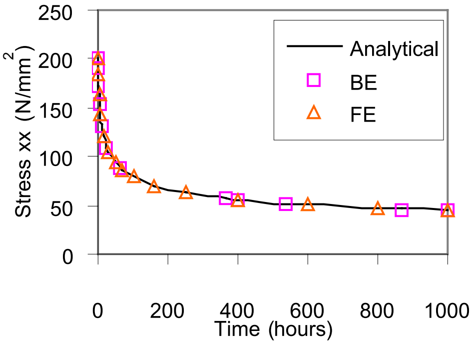

Figure 4 TEST2: creep stress, plane stress, secondary creep.

2. TEST3 and TEST4. The square plate is subjected to biaxial constant tensile stresses of 200 N/mm^2 in the x-direction and 100 N/mm^2 in the y-direction for TEST3 and to biaxial displacements of 0.1 mm in the x-direction and 0.05 mm in the y-direction for TEST4. Both tests are plane strain assumption and secondary creep law. The creep strains and stresses are plotted against time and shown in Figure 5 and Figure 6, respectively. The results show very good agreement with the error being less than 1.2%.

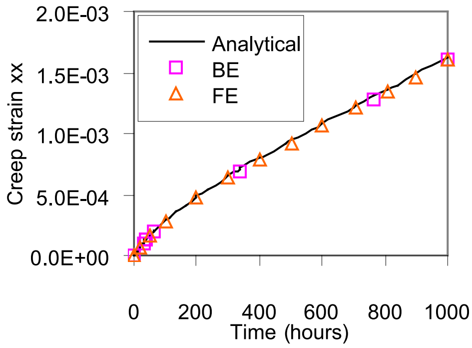

Figure 5 TEST3: creep strain, plane strain, secondary creep.

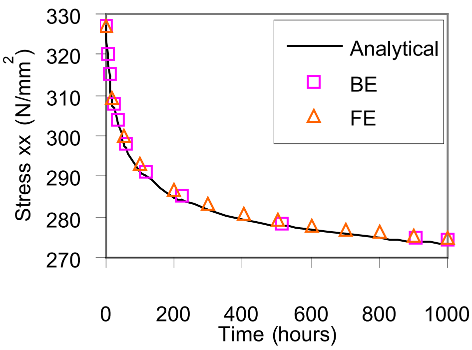

Figure 6 TEST4: creep stress, plane strain, secondary creep.

3. TEST5 and TEST6. The square plate is subjected to a uniaxial constant tensile stress of 200 N/mm^2 in the x-direction for TEST5 and to a uniaxial displacement of 0.1 mm in the y-direction for TEST6. Both tests are plane stress assumption and primary creep law. The creep strains and stresses are plotted against time and shown in Figure 7 and Figure 8, respectively. The results show very good agreement with the error being less than 1.4%. Note that the creep strain solution of FE gives an error up to 10%.

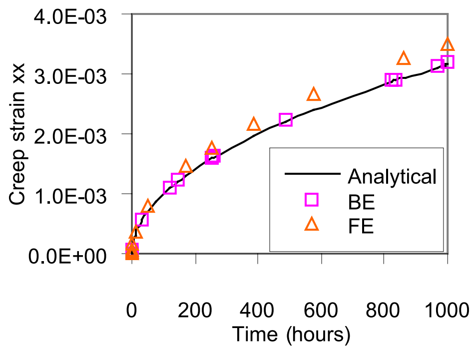

Figure 7 TEST5: creep strain, plane stress, primary creep.

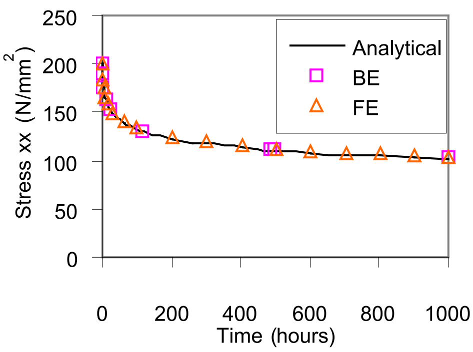

Figure 8 TEST6: creep stress, plane stress, primary creep.

6.2 Square plate with a circular hole

A square plate with a circular hole at the center is analysed. Because of symmetry, only a quarter of the plate is used. The quarter of the plate with a circular hole has the dimensions of 10 mm x 10 mm with a hole of a radius of 3 mm. The boundary and domain are discretised into 28 boundary elements and 48 cells, respectively, as shown in Figure 9. The boundary conditions are as follows:

$u_y = 0$ along line ab.

$u_x = 0$ along line de.

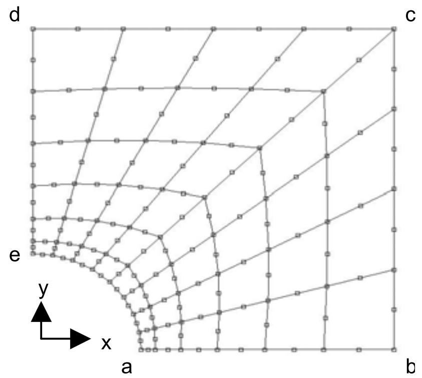

Figure 9 BE and FE mesh for the square plate with a circular hole (28 boundary elements and 48 cells).

The material properties and the creep parameters are the same as those used in the square plate tests. The plate is subjected to a constant tensile stress of 10 N/mm^2 in the x-direction for TEST7 and to a constant displacement of 0.005 mm in the x-direction for TEST8. Plane stress and primary creep law are used. The stresses in the x-direction at 1,000 hours are plotted along the root of the plate and shown in Figure 10 and Figure 11. The results agree well with the FE solutions,

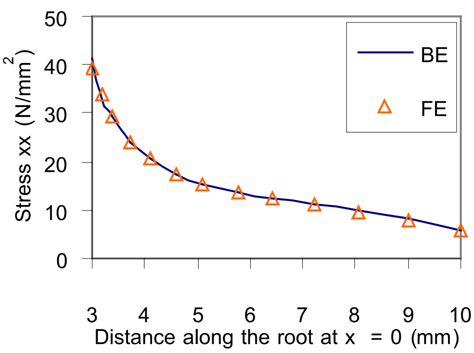

Figure 10 TEST7: creep stress of the square plate with a circular hole under a uniaxial stress load.

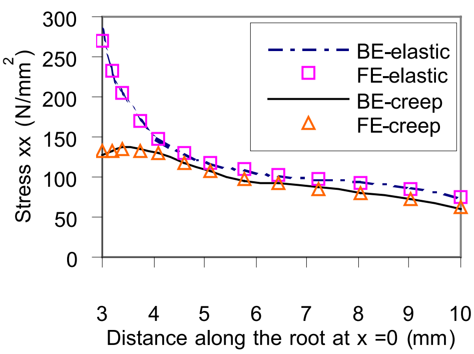

Figure 11 TEST8: creep stress of the square plate with a circular hole under a uniaxial displacement load.

7. Summary

A 2D BE formulation for creep problems using isoparametric quadratic elements is presented. A computer program based on the BE formulation with automatic time stepping and automatic convergence checks is successfully applied to solve the problems of square plates and a plate with a circular hole. The results are compared with the analytical solutions and the FE solutions using MSC.Marc and show good agreement.

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