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Free Vibration Analysis of Symmetrically Laminated Composite Square Plate Using the Extended Kantorovich Method

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Abstract

The free vibration of symmetrically laminated composite square plate with various boundary conditions is analyzed by the extended Kantorovich method. The extended Kantorovich method is an approximate method which uses the iterative calculation to evaluate the natural frequency and to force the final solution to satisfy the boundary conditions. The beam function, the polynomial function and the constant function are applied to the iterative calculation to compare the convergence of the final solution. The Finite Element Method is compared with each other to verify the accuracy. By the result found that the extended Kantorovich method and the Finite Element Method have good agreement and the convergence of the final solution of the beam function is faster than the other function.

***Keywords**:* Vibration, Laminate, Plate, Kantorovich

1. Introduction

Composite materials for plate structures are increasing in the usage due to their high strength, low weight, good fatigue, good corrosion resistance and the properties meet the requirements of specific design.

The extended Kantorovich method is an approximate method which uses to reduce the partial differential equations to the ordinary differential equations. The iterative calculation is used to evaluate the natural frequencies and to force the final solution to satisfy the boundary conditions. The extended Kantorovich method has been reviewed by several researchers. For example, Sakata *et al* [2] used an initial trial function which satisfied the boundary conditions along one direction to evaluate the natural frequency of an orthotropic rectangular plate with various boundary conditions. It was found that the convergence of the final solution was rapid and the particular natural frequency can be obtained separately with good accuracy. Dalaei and Kerr [3], Bercin [5] used an initial trial function which satisfied the boundary conditions along the *y* coordinate direction to derive a closed-form approximate solution for the natural frequency of an orthotropic rectangular clamped plate. The result showed that the final solution was obtained from the fourth iteration and it was independent of an initial trial function. Rajalingham *et al* [4] improved the convergence of the natural frequency of an isotropic rectangular clamped plate. It discovered these shape functions obtained from the extended Kantorovich method can enhance the effectiveness of the Rayleigh-Ritz method. Lee JM *et al* [6] used the beam function as an initial trial function to derive the free vibration of symmetrically laminated composite rectangular plate with all edges elastically restrained against rotation based on the first order anisotropic shear deformation plate theory. The result indicated that the extended Kantorovich method can apply to the free vibration of symmetrically laminated composite with cross-ply rectangular plate but cannot apply to the free vibration of symmetrically laminated composite with angle-ply rectangular plate. Rajalingham *et al* [7] used a plate characteristics function as an initial trial function to derive a closed-form approximate solution for the natural frequency of an isotropic rectangular clamped plate. It was found that the modal parameters were suitable to evaluate for higher natural frequency. Ungbhakorn and Singhatanadgid [9] used an arbitrary function as an initial trial function to evaluate the critical buckling load of symmetrically laminated composite with unidirectional 0° and cross-ply rectangular plate with various boundary conditions. The result showed that the final solution was automatically forced to satisfy the boundary conditions and the critical buckling load was obtained from the fourth iteration. The purpose of this study is to evaluate the natural frequency of symmetrically laminated composite square plate with various boundary conditions by the extended Kantorovich method.

2. Derivation of the iterative differential equations

Hamiton's principle assumes that the system under consideration is characterized by two energy functions, the kinetic energy and the potential energy [8].

$$\delta\int_{t_1}^{t_2}[K-(V+U)]dt=0 \tag{1}$$

where K is the kinetic energy, $V+U$ are the potential energy

The potential energy and the kinetic energy of the symmetrically laminated composite plate, as Fig. 1, can be written as

$$\begin{aligned}\delta\int_{t_1}^{t_2}\Bigg\{&\frac{1}{2}\int_0^a\int_0^b\Bigg[D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2D_{12}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\\&+4D_{16}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x\partial y}\right)+D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2\\&+4D_{26}\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x\partial y}\right)+4D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Bigg]dx\,dy\\&-\frac{1}{2}\int_0^a\int_0^b m(\omega w)^2dx\,dy\Bigg\}dt=0\end{aligned} \tag{2}$$

where D_{ij} is the bending stiffness of composite plate, w is the lateral deflection, m is mass per unit area of plate and ω is the natural circular frequency.

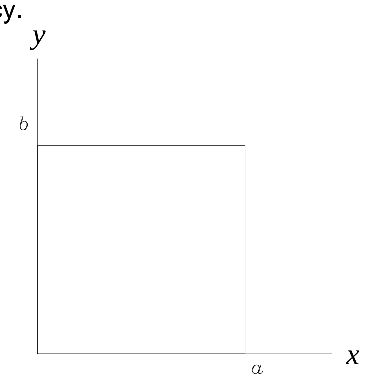

Fig. 1 Rectangular plate

Assume the solution as

$$w(x,y) = X(x)Y(y) \tag{3}$$

substitute equation (3) into equation (2)

$$\frac{1}{2}\int_0^a\int_0^b\left[D_{11}\left(\frac{\partial^2 X}{\partial x^2}Y\right)^2 + 2D_{12}\left(\frac{\partial^2 X}{\partial x^2}Y\right)\left(X\frac{\partial^2 Y}{\partial y^2}\right)\right.$$
$$+D_{22}\left(X\frac{\partial^2 Y}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right)^2 + 4D_{16}\left(\frac{\partial^2 X}{\partial x^2}Y\right)$$
$$\left.+4D_{26}\left(X\frac{\partial^2 Y}{\partial y^2}\right)\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right)\right]dxdy$$
$$-\frac{1}{2}\int_0^a\int_0^b\left[mX^2Y^2\omega^2\right]dx\,dy = 0 \tag{4}$$

If $X(x)$ is defined as priori, equation (4) can be rewritten as

$$\frac{1}{2}\int_0^b\left[S_{1x}D_{11}Y^2 + 2S_{2x}D_{12}Y\left(\frac{\partial^2 Y}{\partial y^2}\right) + S_{3x}D_{22}\left(\frac{\partial^2 Y}{\partial y^2}\right)^2\right.$$
$$+4S_{4x}D_{66}\left(\frac{\partial Y}{\partial y}\right)^2 + 4S_{5x}D_{16}Y\left(\frac{\partial Y}{\partial y}\right)$$
$$\left.+4S_{6x}D_{26}\left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial^2 Y}{\partial y^2}\right)\right]dy$$
$$-\frac{1}{2}\int_b^b S_{3x}mY^2\omega^2 dy = 0 \tag{5}$$

where

$$S_{1x} = \int_0^a\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx, \qquad S_{2x} = \int_0^a\left(X\frac{\partial^2 X}{\partial x^2}\right)dx$$
$$S_{3x} = \int_0^a X^2 dx, \qquad S_{4x} = \int_0^a\left(\frac{\partial X}{\partial x}\right)^2 dx$$
$$S_{5x} = \int_0^a\left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial^2 X}{\partial x^2}\right)dx, \quad S_{6x} = \int_0^a X\left(\frac{\partial X}{\partial x}\right)dx$$

By the variational method and integration by parts equation (5) yields the fourth order ordinary differential equations as equation (6) and the boundary conditions along $y=0$ and $y=b$ as equation (7) and (8)

$$S_{3x}D_{22}\frac{d^4Y}{dy^4} + (2S_{2x}D_{12} - 4S_{4x}D_{66})\frac{d^2Y}{dy^2}$$
$$+(S_{1x}D_{11} - S_{3x}m\omega^2)Y = 0 \tag{6}$$
$$V_y = S_{3x}D_{22}\frac{d^3Y}{dy^3} + (S_{2x}D_{12} - 4S_{4x}D_{66})\frac{dY}{dy}$$
$$-2S_{5x}D_{16}Y \tag{7}$$
$$M_y = S_{3x}D_{22}\frac{d^2Y}{dy^2} + 2S_{6x}D_{26}\frac{dY}{dy}$$
$$+S_{2x}D_{12}Y \tag{8}$$

Similarly when $Y(y)$ is defined as priori, can be written the fourth order ordinary differential equations as equation (9) and the boundary conditions along $x=0$ and $x=a$ as equation (10) and (11)

$$S_{3y}D_{11}\frac{d^4X}{dx^4} + (2S_{2y}D_{12} - 4S_{4y}D_{66})\frac{d^2X}{dx^2}$$
$$+(S_{1y}D_{22} - S_{3y}m\omega^2)X = 0 \tag{9}$$
$$V_x = S_{3y}D_{11}\frac{d^3X}{dx^3} + (S_{2y}D_{12} - 4S_{4y}D_{66})\frac{dX}{dx}$$
$$-2S_{5y}D_{26}X \tag{10}$$
$$M_x = S_{3y}D_{11}\frac{d^2X}{dx^2} + 2S_{6y}D_{16}\frac{dX}{dx}D_{12}X$$
$$+S_{2y}D_{12}X \tag{11}$$

where

$$S_{1y} = \int_0^b\left(\frac{\partial^2 Y}{\partial y^2}\right)^2 dy, \qquad S_{2y} = \int_0^b\left(Y\frac{\partial^2 Y}{\partial y^2}\right)dy$$
$$S_{3y} = \int_0^b Y^2 dy, \qquad S_{4y} = \int_0^b\left(\frac{\partial Y}{\partial y}\right)^2 dy$$
$$S_{5y} = \int_0^b\left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial^2 Y}{\partial y^2}\right)dy, \quad S_{6y} = \int_0^b Y\left(\frac{\partial Y}{\partial y}\right)dy$$

3. Solution of the iterative differential equations

The fourth order ordinary differential in equation (6) can be rewritten in a simple form as

$$\frac{d^4Y}{dy^4}+\left(\frac{2S_{2x}D_{12}-4S_{4x}D_{66}}{S_{3x}D_{22}}\right)\frac{d^2Y}{dy^2}+\left(\frac{S_{1x}D_{11}-S_{3x}m\omega^2}{S_{3x}D_{22}}\right)Y=0$$

consider a case $S_{1x}D_{11}<S_{3x}\,m\omega^2$, the solution can be written as

$$Y(y)=C_{1y}\sin(q_1y)+C_{2y}\cos(q_1y)+C_{3y}\sinh(q_2y)+C_{4y}\cosh(q_2y) \quad (12)$$

where q_1 and q_2 are modal parameters in y coordinate direction, and

$$q_1^2-q_2^2=\frac{2S_{2x}D_{12}-4S_{4x}D_{66}}{S_{3x}D_{22}} \quad (13)$$

$$q_1^2\,q_2^2=\frac{S_{3x}m\omega^2-S_{1x}D_{11}}{S_{3x}D_{22}} \quad (14)$$

substitute equation (12) into the boundary conditions yields the non-trivial solution.

Similarly, the fourth order ordinary differential in equation (9) can be rewritten in a simple form as

$$\frac{d^4X}{dx^4}+\left(\frac{2S_{2y}D_{12}-4S_{4y}D_{66}}{S_{3y}D_{11}}\right)\frac{d^2X}{dx^2}+\left(\frac{S_{1y}D_{22}-S_{3y}m\omega^2}{S_{3y}D_{11}}\right)X=0$$

consider a case $S_{1y}D_{22}<S_{3y}\,m\omega^2$, the solution can be written as

$$X(x)=C_{1x}\sin(p_1x)+C_{2x}\cos(p_1x)+C_{3x}\sinh(p_2x)+C_{4x}\cosh(p_2x) \quad (15)$$

where p_1 and p_2 are modal parameters in x coordinate direction, and

$$p_1^2-p_2^2=\frac{2S_{2y}D_{12}-4S_{4y}D_{66}}{S_{3y}D_{11}} \quad (16)$$

$$p_1^2\,p_2^2=\frac{S_{3y}m\omega^2-S_{1y}D_{22}}{S_{3y}D_{11}} \quad (17)$$

substitute equation (15) into the boundary conditions yields the non-trivial solution.

4. Iterative calculation procedures

1. The iterative calculation can start with choosing an initial trial function in x or y coordinates direction, the following procedure chooses a function in x coordinate direction as an initial trial function.
2. Calculate S_{1x} to S_{6x} by an initial trial function from step 1.
3. Substitute q_2 as a function of q_1 or substitute q_1 as a function of q_2 from relationship equation (13), and then find Eigenvalue q_1 or q_2 and Eigenvector in y coordinate direction.
4. Calculate S_{1y} to S_{6y} by Eigenvector in y coordinate direction obtained from step 3.
5. Substitute p_2 as a function of p_1 or substitute p_1 as a function of p_2 from relationship equation (16), and then find Eigenvalue p_1 or p_2 and Eigenvector in x coordinate direction.
6. Calculate S_{1x} to S_{6x} by Eigenvector in x coordinate direction obtained from step 5.
7. Substitute q_2 as a function of q_1 or substitute q_1 as a function of q_2 from relationship equation (13), and then find Eigenvalue q_1 or q_2 and Eigenvector in y coordinate direction.
8. Compare q_1 and q_2 from step 7 and step 3. If the difference satisfies the specified tolerance, the last q_1 and q_2 are taken as the final

Table 1 The first natural frequency of the $[0/90]_s$ laminates square plate.

(1) Boundary condition CCCC

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	102.132	89.094	40.824	101.558
2	102.125	102.135	102.156	
3	102.125	102.125	102.125	
4		102.125	102.125	

(2) Boundary condition CCCS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	97.523	83.596	28.133	96.927
2	97.517	97.528	97.557	
3	97.517	97.477	97.607	
4		97.517	97.517	

(3) Boundary condition CCSS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	72.252	37.063	28.133	71.949
2	72.242	72.254	72.288	
3	72.242	72.242	72.242	
4		72.242	72.242	

(4) Boundary condition CFCC

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	91.175	74.426	40.206	90.605
2	91.175	91.321	106.374	
3		91.175	91.208	
4		91.175	91.175	

(5) Boundary condition CFCF

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	89.958	74.428	40.824	89.405
2	89.958	90.008	110.156	
3		89.958	90.044	
4		89.958	89.958	

(6) Boundary condition CFCS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	90.699	91.382	28.133	90.132
2	90.699	101.299	101.182	
3		90.717	90.722	
4		90.699	90.699	

Table 1 The first natural frequency of the $[0/90]_s$ laminates square plate (Continue).

(7) Boundary condition CFSC

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	63.653	51.969	40.206	63.404
2	63.653	83.181	83.419	
3		63.704	63.709	
4		63.653	63.653	

(8) Boundary condition CFSF

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	61.982	56.333	40.824	61.771
2	61.982	87.703	87.897	
3		62.125	62.131	
4		61.982	61.982	

(9) Boundary condition CFSS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	62.986	19.751	28.133	62.742
2	62.986	63.007	76.799	
3		62.986	63.028	
4		62.986	62.986	

(10) Boundary condition CSCS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	94.664	79.948	18.009	94.059
2	94.659	94.659	94.659	
3	94.659	94.659	94.659	
4				

(11) Boundary condition CSSS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	68.370	64.583	62.004	68.072
2	68.370	68.372	68.379	
3		68.370	68.370	
4		68.370	68.370	

(12) Boundary condition FSCS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	26.143	19.481	14.139	26.063
2	26.143	26.409	26.269	
3		26.143	26.143	
4		26.143	26.143	

Table 1 The first natural frequency of the $[0/90]_s$ laminates square plate (Continue).

(13) Boundary condition FSFS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	17.991	8.155	0.052	17.986
2	17.991	18.024	18.009	
3		17.991	17.991	
4		17.991	17.991	

(14) Boundary condition SSFS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	20.817	11.786	0.062	20.752
2	20.817	20.829	20.863	
3		20.817	20.817	
4		20.817	20.817	

(15) Boundary condition SSSS

Iteration no.	The extended Kantorovich method			Finite Element Method
	Beam function	Polynomial function	Constant function	
1	48.313	29.510	18.009	48.144
2	48.313	48.313	48.313	
3		48.313	48.313	
4				

Note: The beam function $CC = [\sin(p_1 x) - \sinh(p_1 x)] + \left[\frac{\cos(p_1 a) - \cosh(p_1 a)}{\sin(p_1 a) + \sinh(p_1 a)}\right][\cos(p_1 x) - \cosh(p_1 x)]$

$$CS = [\sin(p_1 x) - \sinh(p_1 x)] - \left[\frac{\sin(p_1 a) - \sinh(p_1 a)}{\cos(p_1 a) - \cosh(p_1 a)}\right][\cos(p_1 x) - \cosh(p_1 x)]$$

$$FC = [\sin(p_1 x) + \sinh(p_1 x)] - \left[\frac{\sin(p_1 a) + \sinh(p_1 a)}{\cos(p_1 a) + \cosh(p_1 a)}\right][\cos(p_1 x) + \cosh(p_1 x)]$$

$$FF = [\sin(p_1 x) + \sinh(p_1 x)] + \left[\frac{\cos(p_1 a) - \cosh(p_1 a)}{\sin(p_1 a) + \sinh(p_1 a)}\right][\cos(p_1 x) + \cosh(p_1 x)]$$

$$FS = [\sin(p_1 x) + \sinh(p_1 x)] - \left[\frac{\sin(p_1 a) + \sinh(p_1 a)}{\cos(p_1 a) + \cosh(p_1 a)}\right][\cos(p_1 x) + \cosh(p_1 x)]$$

$$SS = \sin(p_1 x)$$

$$SF = \sin(p_1 x) + \left[\frac{\sin(p_1 a)}{\sinh(p_1 a)}\right]\sinh(p_1 x)$$

The polynomial function = x^2

The constant function = 1

solution. Otherwise continue the iterative calculation by repeating step 2 to step 7.

9. Calculate the natural frequency from equation (14) or (17).

5. Numerical verification and accuracy

The accuracy of the presented method is verified by comparison the first natural frequency of the symmetrically laminated composite square plate with cross-ply fibers, $[0/90]_s$, with the Finite Element Method, as illustrated in Table 1. Mechanical properties of Kevlar 49 and plate dimension are E_1 = 138 GPa, E_2 = 8.96 GPa, G_{12} = 7.1 GPa, G_{23} = 2.82 GPa, ν_{12} = 0.3, ν_{23} = 0.59, mass per unit volume = 1600 kg/m^3, a = 1 m and *thickness* = *2.5 mm.* The boundary conditions of plate are CCCC, CCCS, CCSS, CFCC, CFCF, CFCS, CFSC, CFSF, CFSS, CSCS, CSSS, FSCS, FSFS, SSFS and SSSS. The first and third letter mean the boundary conditions along $x = 0$ and $x = a$ respectively, the second and fourth letter mean the boundary conditions along $y = 0$ and $y = b$ respectively. The letter “C”, “S” and “F” mean the clamped, simply supported and free boundary conditions respectively. The beam function which satisfied the boundary conditions, the polynomial function and the constant function which unsatisfied the boundary conditions are applied in the iterative calculation.

6. Discussion and conclusion

The free vibration of symmetrically laminated composite square plate with various boundary conditions is analyzed by the extended Kantorovich method. The extended Kantorovich method is an approximate method which uses the iterative calculation to evaluate the natural frequencies and to force the final solution to satisfy the boundary condition. The beam function, the polynomial function and the constant function are used as an initial trial function in the iterative calculation. By the result found that the extended Kantorovich method and the Finite Element Method have good agreement and the convergence of the final solution of the beam function is faster than the other. This indicates that an initial trial function affects the convergence of the final solution and the extended Kantorovich method can use an arbitrary function as an initial trial function.

7. Acknowledgement

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8. References

[1] Jones RM (1975). Mechanics of composite materials, Washington DC, Scripta Book Co.

[2] Sakata T, Takahashi K and Bhat RB (1996). Natural frequency of orthotropic rectangular plates obtained by iterative reduction of the partial differential equation, *Journal of Sound and Vibration*, vol.189(1), pp. 89-101.

[3] Dalaei M and Kerr AD (1996). Natural vibration analysis of clamped rectangular orthotropic plates, *Journal of Sound and Vibration*, vol.189(3), pp. 399-406.

[4] Rajalingham J, Bhat RB and Xistris GD (1996). Vibration of rectangular plates using plate characteristic functions as shape functions in the Rayleigh-Ritz method, *Journal of Sound and Vibration*, vol.193(2), pp. 497-509.

[5] Bercin AN (1996). Free vibration solution for clamped orthotropic plates using the Kantorovich method, *Journal of Sound and Vibration*, vol.196(2), pp. 243-7.

[6] Lee JM, Chung JH and Chung TY (1997). Free vibration analysis of symmetrically laminated composite rectangular plates, *Journal of Sound and Vibration*, vol.199(1), pp. 71-85.

[7] Rajalingham J, Bhat RB and Xistris GD (1997). Vibration of rectangular plates by reduction of the plate partial differential equation into simultaneous ordinary differential equations, *Journal of Sound and Vibration*, vol.203(1), pp. 169-80.

[8] Reddy JN (2004). Mechanics of laminated composite plates and shells theory and analysis, Boca Raton, CRC Press.

[9] Ungbhakorn V and Singhatanadgid P (2006). Buckling analysis of symmetrically laminated composite plates by the extended Kantorovich method, Composite Structures, vol.73(1), pp. 120-8.