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Evaluation of Two Finite Element Schemes for Conjugate Heat Transfer Problems

Atipong Malatip Niphon Wansophark
Pramote Dechaumphai*
Department of Mechanical Engineering, Faculty of Engineering, Chulalongkorn University,
Bangkok, 10330, Thailand

Abstract

This paper presents two finite element schemes for solving conjugate heat transfer problems, where heat conduction in a solid is coupled with heat convection in viscous fluid flow. For solving viscous incompressible thermal flow in fluid region, the Streamline Upwind Finite Element method and the Streamline Upwind Petrov-Galerkin method are selected, while heat conduction in solid region is solved using the standard Galerkin method. The methods use the three-node triangular element with equal-order interpolation functions for all the variables of the velocity components, the pressure and the temperature. The main advantage of the presented approach is to consistently couple heat transfer along the solid-fluid interface. Three test cases, conjugate Couette flow problem in parallel plate channel, counter-flow in heat exchanger, and conjugate natural convection in a square cavity with a conducting wall, are selected to evaluate the presented algorithms.

1. Introduction

Conjugate heat transfer problems are encountered in many practical applications, where heat conduction in solid region is closely coupled with heat convection in an adjacent fluid. There are many engineering areas where conjugate heat transfer should be considered such as heat transfer enhancement with a finned surface, design of thermal insulation, cooling of nuclear reactor, design of solar collector, etc. Most of the computational studies in this research area, however, are based on finite difference and finite volume methods [1]. Numerous publications with computational results show that these methods can perform very well on the problems of interest, but some assumptions on heat transfer coefficients are needed to compute the temperatures along the solid-fluid interface. Furthermore, the determination of unknown temperatures and heat fluxes at the solid-fluid interface is done in an iterative way, usually through the use of the artificial heat transfer coefficient.

For the finite element method, some researchers proposed computational procedure for conjugate heat transfer problems. Misra and Sarkar [2] use the standard Galerkin formulation and solve the continuity, momentum and energy equations simultaneously. Cesini and Paroncini [3] use the streamfunction-vorticity formulation with segregated solution algorithm.

In this paper, two finite element schemes known as the Streamline Upwind Finite Element method [4] and Streamline Upwind Petrov-Galerkin method [5-6] are selected for the analysis of conjugate heat transfer problems. Both methods use equal-order interpolation functions for the velocity components, the pressure and the temperature, and then solved them separately for further improving the computational efficiency. The method also calculates the temperatures and the heat fluxes along the solid-fluid interface directly without the use of the assumed heat transfer coefficient.

2. Theoretical formulation and solution procedure

2.1 Governing equations

The governing equations for conjugate heat transfer problems consist of the conservation of mass or the continuity

* Corresponding author. Tel.: 02-218-6621; Fax: 02-218-6621.
E-mail address: fmepdc@eng.chula.ac.th (P. Dechaumphai)

equation, the conservation of momentums in x and y directions and the conservation of energy.

Continuity equation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1a}$$

Momentum equations,

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1b}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho f_y \tag{1c}$$

where $f_y=-g_y\left[1-\beta\left(T-T_0\right)\right]$.

Energy equation,

$$\rho c\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\rho Q \tag{1d}$$

where u and v are the velocity components in the x and y direction, respectively; ρ is the density, p is the pressure, μ is the viscosity, g_y is the gravitational acceleration constant, β is the volumetric coefficient of thermal expansion, T_o is the reference temperature for which buoyant force in the *y*-direction vanishes, c is specific heat, k is the coefficient of thermal conductivity and Q is the internal heat generation rate per unit volume. Equation (1d) can also be used for solving heat conduction in solid by setting both the velocity components, u and v, as zero.

2.2 Finite element formulations

2.2.1 Streamline Upwind Finite Element method

For the Streamline Upwinding Finite Element formulation, a special treatment for the convection terms is incorporated. These terms are approximated by a monotone streamline upwinding formulation to be used with the triangular element [4]. In this approach, the convection term is first written in the streamline coordinates as,

$$U_s\frac{\partial \phi}{\partial s} \tag{2}$$

where U_s and $\partial/\partial s$ are the velocity and the gradient along the streamline direction, respectively. These terms are evaluated by a streamline tracing method, which keeps track of the direction of the flow within the element.

2.2.2 Streamline Upwind Petrov-Galerkin method

In Streamline Upwind Petrov-Galerkin method, a modified weighting function, W_i, is applied to the convection terms for suppressing the non-physical spatial oscillation in the numerical solution. The modified weighting function is given by Zienkiewicz [6],

$$W_i = N_i+\frac{\alpha h}{2}\frac{\left[u\left(\frac{\partial N_i}{\partial x}\right)+v\left(\frac{\partial N_i}{\partial y}\right)\right]}{|U|} \tag{3}$$

where α is calculated for each element from,

$$\alpha=\alpha_{opt} = coth\ Pe-\frac{1}{Pe} \tag{4a}$$

with

$$Pe=\frac{|U|h}{2k} \text{ and } |U|=\sqrt{u^2+v^2} \tag{4b}$$

where Pe is the Peclet numbers and h is the element size.

The basic idea of both the solution algorithms presented in this paper is to use the two momentum equations for solving the velocity components, use the continuity equation for solving the pressure, and use the energy equation for solving the temperature in solid and fluid regions.

2.2.3 Discretization of momentum and energy equations

The three-nodes triangular element is used in this study. The element assumes linear interpolation functions for the velocity components, the pressure, and the temperature as

$$\phi(x,y)=\sum_{i=1}^{3}N_i(x,y)\phi_i \tag{5}$$

where ϕ is transport property (u, v, p and T) and N_i are the element interpolation functions.

To derive the momentum and the energy equations that correspond to the Streamline Upwind Finite Element scheme and the Streamline Upwind Petrov-Galerkin scheme, the Galerkin method of weighted residuals is employed by multiplying Eqs. (1b-d) with the weighting function, N_i, except for the convection terms which the special treatment as described in the above sections is used. Integration by parts are then performed using the Gauss theorem to yield the element equations in the form,

Momentum equations,

$$[A]\{u\}=\{R_{px}\}+\{R_u\} \tag{6a}$$

$$[A]\{v\}=\{R_{py}\}+\{R_v\}+\{R_{gy}\} \tag{6b}$$

Energy equation,

$$\left[A^T\right]\{T\} = \{R^T\} + \{R_Q^T\} \qquad (7)$$

where the coefficient matrices $[A]$ and $\left[A^T\right]$ contain the known contributions from the convection and diffusion terms. Details of these matrices can be found in ref [4].

2.2.4 Discretization of pressure equation

To derive the pressure equation, the method of weighted residuals is applied to the continuity equation, Eq. (1a). Because the pressure term does not appear in the continuity equation, the relation between velocity components and pressure are thus required. Such relations can be derived from the momentum equations, Eqs. (6a-b) as,

$$A_{ii} u_i = -\sum_{j \neq i} A_{ij} u_j + f_i^u - \int_\Omega N_i \frac{\partial p}{\partial x} d\Omega \qquad (8a)$$

$$A_{ii} v_i = -\sum_{j \neq i} A_{ij} v_j + f_i^v - \int_\Omega N_i \frac{\partial p}{\partial y} d\Omega \qquad (8b)$$

where f_i^u and f_i^v are the surface integral terms and the source term due to the buoyancy.

By applying Eqs. (8a-b) into the continuity equations, the pressure equations can be written in matrix form with unknowns of the nodal pressure as

$$[K]\{p\} = \{F_u\} + \{F_v\} + \{F_b\} \qquad (9)$$

where the details for these element matrices can also found in ref [4].

The above element equations are assembled to yield the global equations for the velocity components, the temperature and the pressure equations. Appropriated boundary conditions are then applied prior to solving for the new velocity components, temperature and pressure values.

2.2.5 Computational procedure

The computational procedure starts from assuming initial nodal velocity components, pressures, and temperatures. The new nodal temperatures are computed using Eq. (7). The new nodal velocity components and pressures are then computed using Eqs. (6a-b) and (9), respectively. The nodal velocity components are then updated using Eqs. (8a-b) with the computed nodal pressures. This process is continued until the specified convergence criterion is met. Such segregated solution procedure helps reducing the computer storage because the equations for the velocity components, the pressure, and the temperature are solved separately.

3. Results

In this section, three example problems are presented. The first example, conjugate Couette flow problem in parallel plate channel, is chosen to evaluate the finite element formulations and to validate the developed computer programs. The second and the third examples, counter-flow in heat exchanger and conjugate natural convection in a square cavity with a conducting wall, are used to illustrate the capability of the presented schemes in the analysis of conjugate heat transfer problems.

3.1 Conjugate Couette flow problem in parallel plate channel

The first example for evaluating the finite element formulations and validating the developed computer programs is the problem of conjugate Couette flow problem in parallel plate channel. As shown in Fig. 1, the upper wall moves at a constant velocity and the other wall is stationary conducting solid. The other surface of the conducting solid is maintained at a constant temperature that is higher than the constant temperature of the

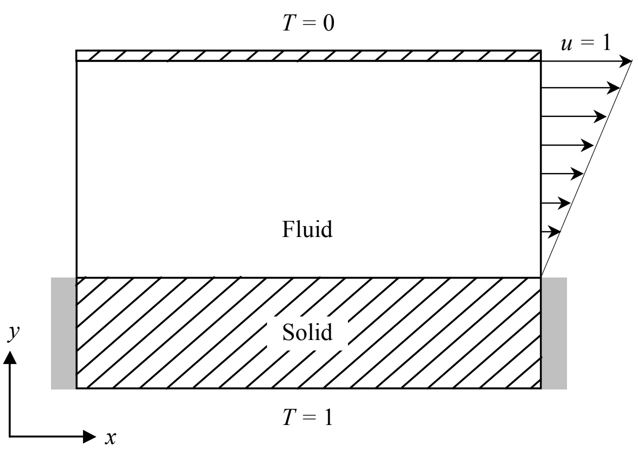


Fig. 1. A conjugate Couette flow problem in parallel plate channel.

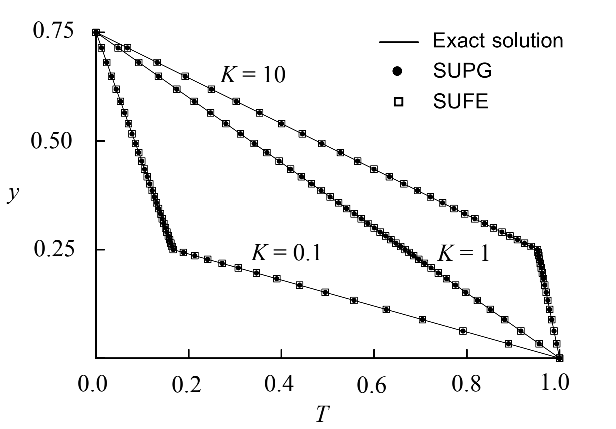


Fig. 2. Comparison of conjugate benchmark solutions for Couette flow problem.

opposing channel wall. The numerical results are compared with the analytical solution as shown in Ref. [7]. Fig. 2 shows that the computational results from both finite element schemes demonstrate excellent agreement with the analytical solution for varying conductivity ratios, $K = k_s / k_f$, where k_s and k_f are solid and fluid heat conduction coefficient respectively. The numerical results of the temperatures from the Streamline Upwind Petrov-Galerkin method and the Streamline Upwind Finite Element method are compared within 0.04% of the analytical solutions.

3.2 Conjugate counter flow heat exchanger

To validate the numerical schemes with the second test example, a conjugate counter flow heat exchanger problem is selected. This heat exchanger consists of two parallel flow passages with widths a_1 and a_3 separated by a solid plate with thickness of a_2 as shown in Fig. 3. The outer walls of the flow passages are assumed to be adiabatic. The same properties and uniform inlet velocity and temperature profiles are assumed for the hot and cold fluids. Parameters adopted in the computation are as follows, geometrical sizes a_1 = a_2 = a_3 = 0.1 and L = 1.0, the flow in upper channel parameters u_1 = 0.2, T_1 = 800, Re = 133.33 and Pr = 0.75, the flow in lower channel parameters u_2 = 0.1, T_2 = 300, Re = 66.67 and Pr = 0.75, conduction ratio, K = 5. The finite element model consisting of 1,763 nodes and 3,360 triangles, as shown in Fig. 4, is used in this study. Fig. 5 shows the predicted temperature contours in entire domain. The predicted temperature distributions at $x = L/2$ from both presented schemes are compared with the results from Chen and Han [8] as shown in Fig. 6. The figure shows good agreement of the solutions.

3.3 Conjugate natural convection in a square cavity with a conducting wall

To further evaluate the effectiveness of the presented schemes, the problem of conjugate natural convection in a square cavity with a conducting wall as shown in Fig. 7, is selected. The cavity is heated at the left side (solid wall) and cooled at the right side, all other boundaries are insulated. Fig. 8 shows the finite element model that consists of 2,009 nodes and 3,840 triangles. Figs. 9 and 10 show the predicted streamline and temperature contours for different thermal conductivity ratios of K = 1 and 10 at Grashof numbers of 10^3 and 10^5, respectively. The temperature and the heat flux distributions along the solid-fluid interface with the variation of conduction ratio, K, are shown in Figs. 11(a) and (b), respectively. Table 1 compares the predicted average Nusselt numbers at interface, $\overline{Nu}_{x=0}$. The computational results are compared with the results from Hribersek [9] which show good agreement of the solutions of average Nusselt numbers for both temperature and heat flux.

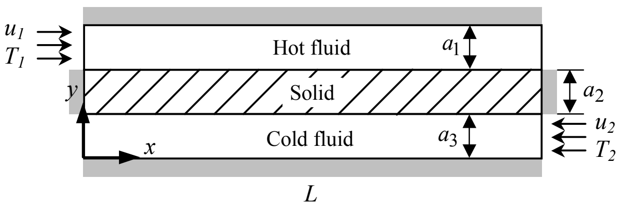


Fig. 3. A conjugate counter flow heat exchanger.

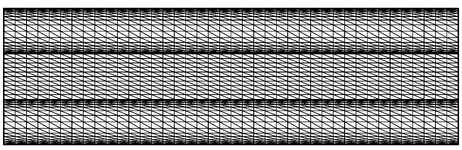


Fig. 4. Finite element model for conjugate counter flow heat exchanger.

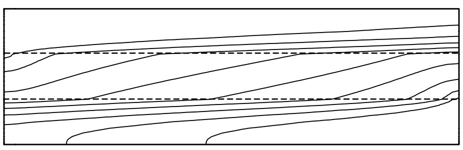


Fig. 5. Predicted temperature contours for a conjugate counter flow heat exchanger.

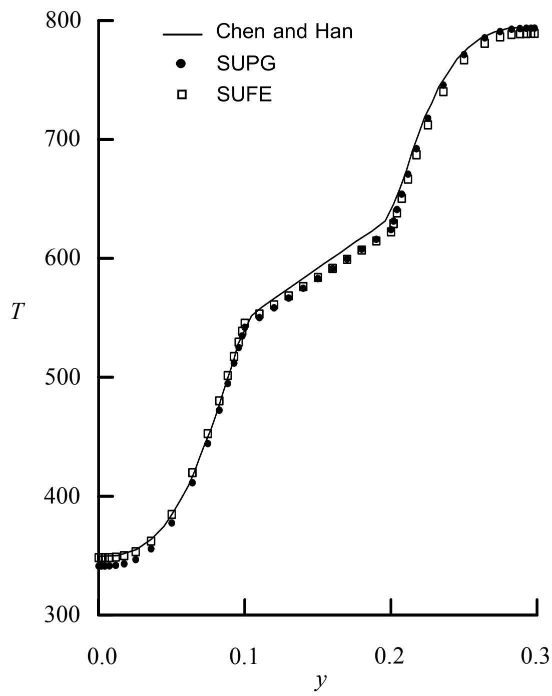


Fig. 6. The temperature profiles at $x = L/2$ for a conjugate counter flow heat exchanger.

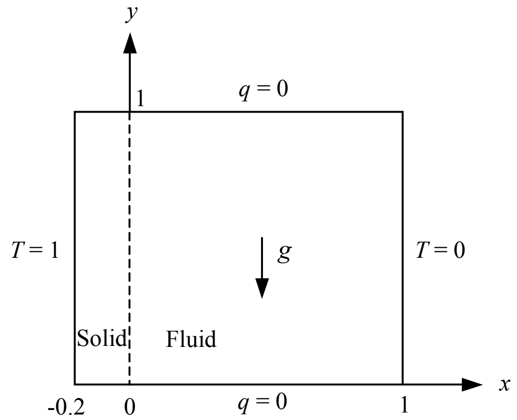


Fig. 7. Conjugate natural convection problem.

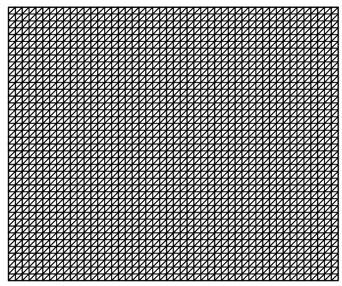
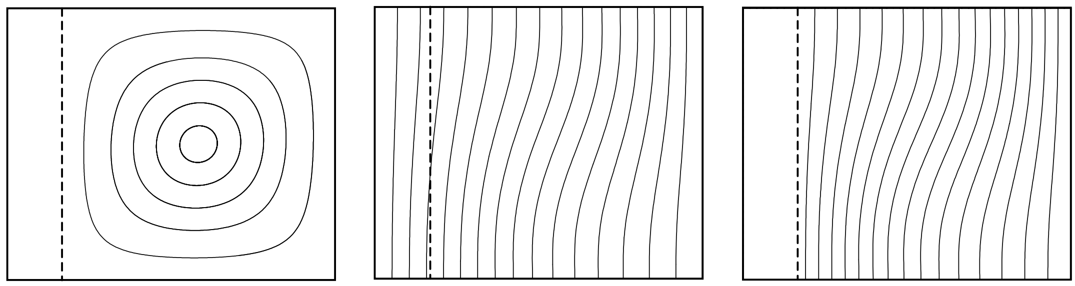
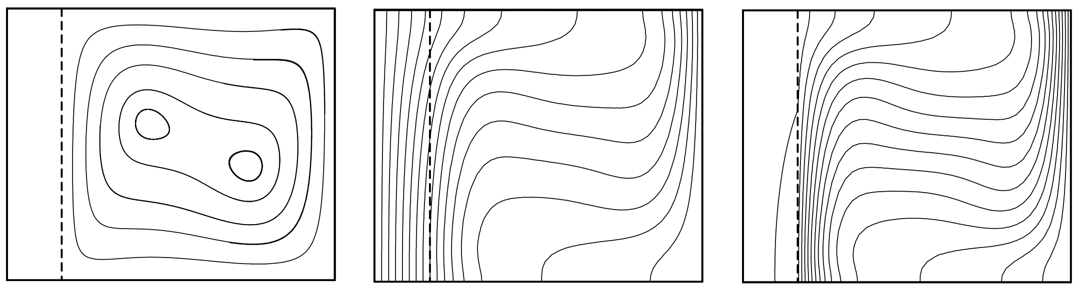


Fig. 8 Finite element model for the conjugate natural convection problem.



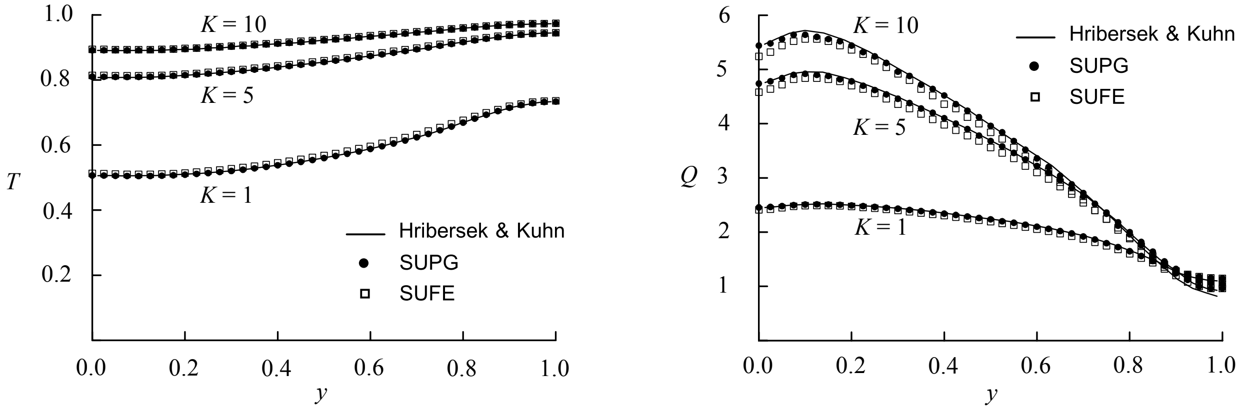
(a) (b) (c)

Fig. 9. (a) Streamline contours for K = 10, (b) Temperature contours for K = 1 and (c) Temperature contours for K = 10, all at Gr = 10^3.



(a) (b) (c)

Fig. 10. (a) Streamline contours for K = 10, (b) Temperature contours for K = 1 and (c) Temperature contours for K = 10, all at Gr = 10^5.



(a) (b)

Fig 11. (a) Interface temperatures and (b) Interface heat fluxes, all at Gr = 10^5.

Table 1 Variation of the overall Nusselt numbers.

Gr		Average Nusselt numbers along interface (% difference from Ref. [9])		
Conductivity ratio $K = k_s / k_f$		1	5	10
10^3	Hribersek [9]	0.87	1.02	1.04
10^3	SUPG	0.87 (0.0%)	1.02 (0.0%)	1.04 (0.0%)
10^3	SUFE	0.85 (2.29%)	1.03 (0.98%)	1.04 (0.0%)
10^5	Hribersek [9]	2.08	3.42	3.72
10^5	SUPG	2.07 (0.48%)	3.39 (0.87%)	3.67 (1.34%)
10^5	SUFE	2.04 (1.92%)	3.30 (3.51%)	3.60 (3.22%)

4. Conclusions

Two finite element methods for conjugate heat transfer problems are presented. The methods use three-node triangular element for the analysis of viscous incompressible thermal flow in the fluid region and heat transfer in the solid region. The convection terms in the momentum and the energy equations are treated by the Streamline Upwind Finite Element method and the Streamline Upwind Petrov-Galerkin method to suppress the non-physical spatial oscillation in the numerical solutions. The corresponding finite element equations are derived and corresponding computer programs have been developed. The test cases highlight the benefit of the finite element method for the analysis of conjugate heat transfer problems that can compute the temperatures along the solid-fluid interface directly.

5. Acknowledgement

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6. References

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