การประยุกต์ใช้ทฤษฎีวิเคราะห์ความคงทนของเสถียรภาพ ในการออกแบบระบบควบคุมเชิงเส้นแบบคงทนสำหรับระบบหลายอินพุท

Using Robust Stability Analysis Theorems for Robust Controller Design of Multiple Input Systems

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บทคัดย่อ

สำหรับระบบควบคุมเชิงเส้นที่มีเสถียรภาพจำนวนมาก ปัญหาที่สำคัญที่สุดประการหนึ่งคือการคำนวณขอบเขตความไม่แน่นอนของพารามิเตอร์ในระบบที่ยอมให้เกิดขึ้นได้ ในช่วงเวลาหลายสิบปีที่ผ่านมา พบว่ามีทฤษฎีที่ใช้ในการวิเคราะห์ความคงทนของเสถียรภาพนี้ถูกเสนอไว้เป็นจำนวนมาก จนกระทั่งในปัจจุบัน กลุ่มของทฤษฎีดังกล่าวได้คลอบคลุมความไม่แน่นอนหลากหลายประเภท และขอบเขตความไม่แน่นอนที่ยอมให้เกิดขึ้นได้ก็ขยายกว้างขึ้นอย่างต่อเนื่อง ในช่วงเวลาไม่นานมานี้ ได้มีการเสนอวิธีการที่สามารถขยายการประยุกต์ใช้ทฤษฎีเพื่อการวิเคราะห์ความคงทนของเสถียรภาพประเภทแกมม่าสำหรับการออกแบบระบบควบคุมเชิงเส้นแบบคงทนเฉพาะระบบที่มีจำนวนของสัญญาณอินพุทเป็นหนึ่ง โดยวิธีการดังกล่าวสามารถนำมาซึ่งกฎการควบคุมที่ยังความคงทนของเสถียรภาพให้กับระบบควบคุมภายใต้ขอบเขตความไม่แน่นอนที่กว้างในระดับที่น่าพอใจ บทความวิจัยฉบับนี้นำเสนอวิธีการประยุกต์ใช้ทฤษฎีวิเคราะห์ความคงทนของเสถียรภาพประเภทแกมม่าสำหรับการออกแบบระบบควบคุมเชิงเส้นแบบคงทนในระบบที่มีจำนวนของสัญญาณอินพุทหลายสัญญาณ เมื่อนำไปใช้ทดสอบกับตัวอย่างแบบจำลองทางคณิตศาสตร์ พบว่าขอบเขตความไม่แน่นอนที่ยอมให้เกิดขึ้นได้มีขนาดกว้างในระดับที่น่าพอใจเช่นกัน

Abstract

For many stable linear control systems, a crucial problem is to compute allowable uncertainty bounds on system parameters. For decades, a large number of theorems have been proposed to deal with this robust stability analysis problem. They now cover various classes of uncertainties, and the corresponding allowable bounds are increasingly large. Recently, a technique was proposed for extending applications of robust stability analysis theorems in class "gamma" over robust controller design of single input systems, and it was shown that the resulting control law could stabilize linear systems subjected to nonlinear time-varying uncertainties with satisfactorily large allowable bounds. This paper extends applications of these class gamma theorems over robust controller design of multiple input systems. Numerical examples show that the resulting allowable uncertainty bounds are satisfactorily large as well.

1. Introduction

The problem of finding robust linear controllers for stabilizing linear systems subjected to various classes of nonlinear time-varying uncertainties has been considered in numerous papers. During recent decades, Lyapunov Stability and Riccati equation have been employed to reduce conservatism of allowable uncertainty bounds very effectively. This can be tracked back to [1] and many foregoing papers, but the first paper that uses these theoretical tools to formalize this problem as quadratic stabilization is [2]. For years, the early results have been extended in many successive papers. During this period, [3] and [4] proposed stabilizing techniques for the cases in which uncertainties appear in the system matrix, and in the input matrix respectively. After that, [5] presented a general technique that recognizes the formers as its special cases, while providing original innovative extensions. Despite of this remarkable result, it is applicable to a single class of structured uncertainties only. In addition, the formulation is all algebraic, and thus provides little insights into the solutions other than that they satisfy the Riccati equation.

The problem of robust stability analysis (RSA), in which allowable uncertainty bounds are computed for stable linear control systems, has been considered in parallel with the above problem of robust controller design [6-9]. While it seems that no major result on the problem of robust quadratic stabilization has been reported after the publication of [5], robust stability analysis theorems have been steadily formulated to reduce conservatism of allowable uncertainty bounds. These useful RSA theorems have been accumulated for many years, and they now cover various classes of uncertainties. Motivated by this fact, [10] employed matrix algebra and geometry to propose a new class-gamma RSA theorem, and a technique for extending the uses of all RSA thoerms in class gamma over robust controller design of single-input linear systems. Using these, it was shown in numerical examples that the resulting allowable uncertainty bounds could be less conservative than those resulting from [5]. In this paper, we extend the results in [10] to cover multiple-input linear systems.

2. Definitions and Notations

In this section, the objective is to introduce mathematical notations and terms used in [10] and in our extension. It is assumed that the system of interest is described by:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\,\mathbf{u}(\mathbf{x}) + \mathbf{f}_{\Sigma}(\mathbf{x}, t, \mathbf{u}(\mathbf{x})) \quad (1)$$

where $\mathbf{A} \in \Re^{n\times n}$ and $\mathbf{B} \in \Re^{n\times m}$ are known, $\mathbf{u}(\mathbf{x}) \in \Re^{m}$, and $\mathbf{f}_{\Sigma}(\mathbf{x}, t, \mathbf{u}) \in \Re^{n}$ is the uncertainty vector that is locally Lipschitz in $\mathbf{x}$ and t, and is vanishing at the origin. It is assumed that the state $\mathbf{x} = [x_1 \;\ldots\; x_n]^T$ is available for feedback, and the pair $[\mathbf{A}, \mathbf{B}]$ is controllable or stabilizable. We are interested in nontrivial cases in which $n > 1$ and $m > 0$. When the uncertainty vector is zero, we call Eq. (1) the nominal linear model. It is desired to stabilize the system such that the equilibrium point of interest at the origin is uniformly globally asymptotically stable using the linear state-feedback control:

$$\mathbf{u}(\mathbf{x}) = -\mathbf{K}\mathbf{x} \quad (2)$$

where $\mathbf{K} \in \Re^{m\times n}$ is such that all the eigenvalues of $\bar{\mathbf{A}} \equiv [\mathbf{A} - \mathbf{BK}]$ are in the LHP. This yields:

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \mathbf{f}_\Omega(\mathbf{x}, t) \quad (3)$$

where $\mathbf{f}_\Omega(\mathbf{x}, t) \equiv \mathbf{f}_\Sigma(\mathbf{x}, t, \mathbf{u}(\mathbf{x}))|_{\mathbf{u}(\mathbf{x})=-\mathbf{Kx}}$. Now, we define a positive-definite quadratic Lyapunov function:

$$V(\mathbf{x}) = (1/2)\mathbf{x}^T\mathbf{P}\mathbf{x} \quad (4)$$

where $\mathbf{P}$ is a symmetric positive-definite matrix obtained from the Lyapunov equation:

$$-\mathbf{Q} = (1/2)[\mathbf{P}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{P}] \quad (5)$$

where $\mathbf{Q}$ is a symmetric positive-definite matrix to be specified. The time derivative of the Lyapunov function along trajectories of the nominal linear model is denoted by:

$$\dot{V}_L(\mathbf{x}) = -\mathbf{x}^T\mathbf{Q}\mathbf{x} = F_L(\mathbf{x}) - \mathbf{G}_L(\mathbf{x})\mathbf{K}\mathbf{x} \quad (6)$$

where $F_L(\mathbf{x}) \equiv \mathbf{x}^T\mathbf{M}\mathbf{x}$, $\mathbf{M} \equiv (1/2)[\mathbf{PA} + [\mathbf{PA}]^T] \in \Re^{n\times n}$, and $\mathbf{G}_L(\mathbf{x}) \equiv \mathbf{x}^T\mathbf{PB} \in \Re^{1\times m}$. It is clear that $\dot{V}_L(\mathbf{0}) = 0$. We know from Lyapunov stability that the nominal linear model is stable when $\dot{V}_L(\mathbf{x})$ is globally negative definite. The time derivative of the Lyapunov function along trajectories of the uncertain system is:

$$\begin{aligned} \dot{V}_N(\mathbf{x}, t) &= F_L(\mathbf{x}) - \mathbf{G}_L(\mathbf{x})\mathbf{K}\mathbf{x} + F_\Omega(\mathbf{x}, t) \\ &= \dot{V}_L(\mathbf{x}, t) + F_\Omega(\mathbf{x}, t) \\ &= F_\Delta(\mathbf{x}, t) - \mathbf{G}_L(\mathbf{x})\mathbf{K}\mathbf{x} \end{aligned} \quad (7)$$

where $F_\Omega(\mathbf{x}, t) \equiv \mathbf{x}^T\mathbf{P}\mathbf{f}_\Omega(\mathbf{x}, t)$, $F_\Delta(\mathbf{x}, t) \equiv F_L(\mathbf{x}) + F_\Omega(\mathbf{x}, t)$, and $\dot{V}_N(\mathbf{0}, t) = 0$. We know from Lyapunov stability that the uncertain system is uniformly globally asymptotically stable when $\dot{V}_N(\mathbf{x}, t)$ is uniformly globally negative definite. In our following discussions, we abbreviate "uniformly globally asymptotically stable" with "stable", and denote the maximum eigenvalue of a matrix $\mathbf{\Gamma} \in \Re^{n\times n}$ by $\lambda_{\mathbf{\Gamma}1}$ unless otherwise stated. We abbreviate "hyperplane" with "plane", denote by $S_{\mathbf{\Pi}=\mathbf{0}}$ the set $\{\mathbf{x} \mid \mathbf{\Pi}(\mathbf{x}) = \mathbf{0}\}$ where $\mathbf{\Pi}(\mathbf{x}) \in \Re^I$ and I is an appropriate integer, by $P_{\{\mathbf{x}_i,\ldots,\mathbf{x}_j\}}$ the plane spanned by $\{\mathbf{x}_i,\ldots,\mathbf{x}_j\}$ where $\{\mathbf{x}_i,\ldots,\mathbf{x}_j\} \in \Re^n$, by $R_{[\Theta<0]\cup\{\mathbf{0}\}}$ the set $\{\mathbf{x} \mid \Theta(\mathbf{x}) < 0\} \cup \{\mathbf{0}\}$ where $\Theta(\mathbf{x}) \in \Re$. We often call $S_{\mathbf{\Pi}=\mathbf{0}}$ a zero surface.

A RSA theorem is said to be in class-gamma [10] if it guarantees that $\dot{V}_N(\mathbf{x}, t)$ is uniformly globally negative definite when $\bar{\mathbf{A}}$ is stable and:

$$a_\gamma < 0$$

where $a_\gamma \in \Re$ is a function of $\bar{\mathbf{A}}$, $\mathbf{Q}$, and specifications of the uncertainties, which may be structured or unstructured.

3. A Class-Gamma RSA Theorem

While our controller design technique is not limited to a specific class of uncertainties, we assume that the uncertainties are structured as in [10] to be consistent. The structured uncertainties are given by:

$$\mathbf{f}_\Omega(\mathbf{x}, t) \equiv \textstyle\sum_{j=1}^r [h_j(\mathbf{x}, t)\mathbf{E}_j]\mathbf{x} \quad (8)$$

where $h_j(\mathbf{x}, t) \in [h_{lj}, h_{uj}] \in \Re$ is an unknown function, but with known lower bound $h_{lj} \le 0$ and known upper bound $h_{uj} \ge 0$, and $\mathbf{E}_j \in \Re^{n\times n}$ is known for all j = 1, 2, ..., r. The values of h_{lj}, h_{uj}, and $\mathbf{E}_j$ are called uncertainty specifications. Under Eq.(8), the model is now:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{BK}\mathbf{x} + \textstyle\sum_{j=1}^r [h_j(\mathbf{x}, t)\mathbf{E}_j]\mathbf{x} \\ &= \bar{\mathbf{A}}\mathbf{x} + \textstyle\sum_{j=1}^r [h_j(\mathbf{x}, t)\mathbf{E}_j]\mathbf{x} \end{aligned} \quad (9)$$

In [10], a class-gamma theorem for handling the structured uncertainties in Eq. (9) was primarily proposed for the single-input systems. We now show in the following that the theorem is valid without assuming that m = 1.

Theorem 1: If $\bar{\mathbf{A}} \equiv [\mathbf{A} - \mathbf{BK}]$ is stable, then $\dot{V}_N(\mathbf{x}, t)$ is uniformly globally negative definite when:

$$\lambda_{\mathbf{Z}1} < 0 \quad (10)$$

where $\lambda_{\mathbf{Z}1}$ is the maximum real eigenvalue of $\mathbf{Z} = \mathbf{Z}^T$ obtained by:

1) Specified $\mathbf{Q}$ and $\bar{\mathbf{A}}$ to compute $\mathbf{P}$ from the Lyapunov equation.
2) Compute $\bar{\mathbf{A}}_l = \bar{\mathbf{A}} + \sum_{j=1}^r h_{lj}\mathbf{E}_j$, and $\mathbf{\Phi} = \mathbf{P}\bar{\mathbf{A}}_l + \bar{\mathbf{A}}_l^T\mathbf{P}$.
3) Compute $\mathbf{\Psi}_j = [\mathbf{P}\mathbf{E}_j + \mathbf{E}_j^T\mathbf{P}] = \mathbf{\Psi}_j^T$.
4) Compute $\mathbf{\Lambda}_{\mathbf{\Psi}_j} = \mathbf{T}_{\mathbf{\Psi}_j}^T\mathbf{\Psi}_j\mathbf{T}_{\mathbf{\Psi}_j} = \text{diag}[\lambda_{\mathbf{\Psi}j1} \ \ldots \ \lambda_{\mathbf{\Psi}jn}]$, where $\mathbf{T}_{\mathbf{\Psi}_j} = [\mathbf{v}_{\mathbf{\Psi}j1} | \ldots | \mathbf{v}_{\mathbf{\Psi}jn}]$, and $\{\mathbf{v}_{\mathbf{\Psi}j1}, \ \ldots \ , \mathbf{v}_{\mathbf{\Psi}jn}\}$ is the set of n orthonormal eigenvectors of $\mathbf{\Psi}_j$.
5) Compute $\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}$ by setting all negative elements of $\mathbf{\Lambda}_{\mathbf{\Psi}j}$ to zero
6) Compute $\mathbf{\Psi}_j^{\ge 0} = \mathbf{T}_{\mathbf{\Psi}_j}\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}\mathbf{T}_{\mathbf{\Psi}_j}^T$.
7) Compute $\mathbf{Z} \equiv \mathbf{\Phi} + \sum_{j=1}^r [(h_{uj} - h_{lj})\mathbf{\Psi}_j^{\ge 0}]$.

Proof: We write for $h_j(\mathbf{x}, t)$ *, j = 1, 2, ..., r:*

$$h_j(\mathbf{x}, t) = h_{lj} + h_j(\mathbf{x}, t) - h_{lj} \equiv h_{lj} + l_j(\mathbf{x}, t) \quad (11)$$

where $l_j(\mathbf{x}, t) \equiv h_j(\mathbf{x}, t) - h_{lj}$. Since $h_j(\mathbf{x}, t) \in [h_{lj}, h_{uj}]$, $l_j(\mathbf{x}) \in [0, h_{uj} - h_{lj}]\forall j$. Substituting $h_{lj} + l_j(\mathbf{x})$ for $h_j(\mathbf{x}, t)$ in Eq. (10) yields:

$$\dot{\mathbf{x}} = \bar{\mathbf{A}}_l\mathbf{x} + \textstyle\sum_{j=1}^r l_j(\mathbf{x}, t)\mathbf{E}_j\mathbf{x} \quad (12)$$

Differentiating the Eq. (4) along trajectories of Eq. (12) yields:

$$\dot{V}_N(\mathbf{x}, t) = (1/2)\mathbf{x}^T\mathbf{\Phi}\mathbf{x} + (1/2)\textstyle\sum_{j=1}^r l_j(\mathbf{x}, t)\mathbf{x}^T\mathbf{\Psi}_j\mathbf{x} \quad (13)$$

Since $\mathbf{\Psi}_j^T = \mathbf{\Psi}_j \ \forall j$, $\mathbf{\Psi}_j$ has a set of n real eigenvalues $\{\lambda_{\mathbf{\Psi}j1}, \ \ldots \ , \lambda_{\mathbf{\Psi}jn}\}$ and the corresponding set of n orthonormal eigenvectors $\{\mathbf{v}_{\mathbf{\Psi}j1}, \ \ldots \ , \mathbf{v}_{\mathbf{\Psi}jn}\}$ [11]. Using the linear transformation $\mathbf{x} = \mathbf{T}_{\mathbf{\Psi}_j}\mathbf{z}$, we write:

$$\mathbf{x}^T\mathbf{\Psi}_j\mathbf{x} = \mathbf{z}^T[\mathbf{T}_{\mathbf{\Psi}_j}^T\mathbf{\Psi}_j\mathbf{T}_{\mathbf{\Psi}_j}]\mathbf{z} \equiv \mathbf{z}^T\mathbf{\Lambda}_{\mathbf{\Psi}_j}\mathbf{z} \quad (14)$$

where $\mathbf{T}_{\mathbf{\Psi}_j} = [\mathbf{v}_{\mathbf{\Psi}j1} | \ldots | \mathbf{v}_{\mathbf{\Psi}jn}]$, and $\dot{\mathbf{x}} = \bar{\mathbf{A}}_l\mathbf{x} + \sum_{j=1}^r l_j(\mathbf{x}, t)\mathbf{E}_j\mathbf{x}$. We set all negative elements of $\mathbf{\Lambda}_{\mathbf{\Psi}j}$ to zeros to produce $\mathbf{\Lambda}_{\mathbf{\Psi}j}^{\ge 0}$. Thus, $\mathbf{z}^T[\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}]\mathbf{z} \ge 0$, and $\mathbf{z}^T[\mathbf{\Lambda}_{\mathbf{\Psi}j}^{\ge 0}]\mathbf{z} \ge \mathbf{z}^T\mathbf{\Lambda}_{\mathbf{\Psi}j}\mathbf{z} = \mathbf{x}^T\mathbf{\Psi}_j\mathbf{x}$. It follows that:

$$\mathbf{z}^T[\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}]\mathbf{z} = \mathbf{x}^T[\mathbf{T}_{\mathbf{\Psi}_j}^{-1}]^T[\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}][\mathbf{T}_{\mathbf{\Psi}_j}^{-1}]\mathbf{x} \equiv \mathbf{x}^T\mathbf{\Psi}_j^{\ge 0}\mathbf{x} \ge 0 \quad (15)$$

where $\mathbf{\Psi}_j^{\ge 0} = [\mathbf{T}_{\mathbf{\Psi}_j}^{-1}]^T[\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}][\mathbf{T}_{\mathbf{\Psi}_j}^{-1}]$. Because $\mathbf{T}_{\mathbf{\Psi}_j}$ is orthogonal, we have $\mathbf{T}_{\mathbf{\Psi}_j}^{-1} = \mathbf{T}_{\mathbf{\Psi}_j}^T$, and $\mathbf{\Psi}_j^{\ge 0} = [\mathbf{T}_{\mathbf{\Psi}_j}][\mathbf{\Lambda}_{\mathbf{\Psi}_j}^{\ge 0}][\mathbf{T}_{\mathbf{\Psi}_j}^T]$. Now, because $[\mathbf{\Psi}_j^{\ge 0}]^T = \mathbf{\Psi}_j^{\ge 0}$ and because $(h_{uj} - h_{lj}) \ge l_j(\mathbf{x}) > 0$, it follows that:

$$l_j(\mathbf{x}, t)[\mathbf{x}^T\mathbf{\Psi}_j\mathbf{x}] \le (h_{uj} - h_{lj})[\mathbf{x}^T\mathbf{\Psi}_j^{\ge 0}\mathbf{x}] \ \forall\mathbf{x} \quad (16)$$

Applying the above inequality to Eq. (14) yields:

$$\dot{V}_N(\mathbf{x}, t) \le (1/2)\mathbf{x}^T\mathbf{\Phi}\mathbf{x} + (1/2)\textstyle\sum_{j=1}^r((h_{uj} - h_{lj})[\mathbf{x}^T\mathbf{\Psi}_j^{\ge 0}\mathbf{x}]) \quad (17)$$

It follows from Eq. (17) that $\dot{V}_N(\mathbf{x}, t)$ is uniformly globally negative definite when $\lambda_{\mathbf{Z}1} < 0$. ⊗

4. Properties of Zero Surfaces Associated with the Lyapunov Time Derivative along Trajectories of the Nominal Linear Models with Multiple Inputs

In this section, we extend fundamental properties of zero surfaces associated with $\dot{V}_L(\mathbf{x})$ for single-input systems [10] over multiple input systems. A fundamental property of the zero surface $S_{\mathbf{G}_L=\mathbf{0}}$ is:

Lemma 1: If $\dot{V}_L(\mathbf{x})$ is globally negative definite, then $S_{\mathbf{G}_L=\mathbf{0}} \subset R_{[F_L<0]\cup\{\mathbf{0}\}}$

Proof: Consider the expression for $\dot{V}_L(\mathbf{x})$ in Eq. (6). Because $rank(\mathbf{PB}) = rank(\mathbf{B}) \ge 1$, the zero surface $S_{\mathbf{G}_L=\mathbf{0}}$ exists and is a plane of dimension $n - rank(\mathbf{B}) \le n - 1$. Because $\dot{V}_L(\mathbf{x})$ is globally negative definite, it must be true that $\dot{V}_L(\mathbf{x})$ is negative on $S_{\mathbf{G}_L=\mathbf{0}}$, except at the origin where $\mathbf{G}_L(\mathbf{0}) = \mathbf{0}$ and $V_L(\mathbf{0}, t) = 0$. Because of these and because of the structure of $\dot{V}_L(\mathbf{x})$, we know that the region $R_{[F_L<0]\cup\{\mathbf{0}\}}$ must exist such that $S_{\mathbf{G}_L=\mathbf{0}} \subset R_{[F_L<0]\cup\{\mathbf{0}\}}$. ⊗

In addition to the property in Lemma 1, we examine geometrical properties of the function $F_L(\mathbf{x})$, which are completely determined by $\mathbf{M}$. This is given in Theorem 2:

Theorem 2: When $\mathbf{A}$ is unstable or marginally stable, the maximum eigenvalue of $\mathbf{M}$ is non-negative, and the number of negative eigenvalues of $\mathbf{M}$ is at least $n - rank(\mathbf{B})$.

Proof: Because $\mathbf{M} = \mathbf{M}^T \in \Re^{n\times n}$, $\mathbf{M}$ has a set of n real eigenvalues $\boldsymbol{\lambda}_\mathbf{M}$ and a set of the corresponding n real orthonormal eigenvectors $V_\mathbf{M}$, where $\boldsymbol{\lambda}_\mathbf{M} = \{\lambda_{\mathbf{M}1}, \ldots, \lambda_{\mathbf{M}n}\}$ and $V_\mathbf{M} = \{\mathbf{v}_{\mathbf{M}1}, \ldots, \mathbf{v}_{\mathbf{M}n}\}$ with $\mathbf{v}_{\mathbf{M}i}^T \mathbf{v}_{\mathbf{M}i} = 1$, $i = 1, \ldots, n$ respectively. We can employ $V_\mathbf{M}$ as a basis set for generating $\Re^n$. Now, when $\mathbf{A}$ is unstable or marginally stable, $F_L(\mathbf{x})$ cannot be globally negative definite. Otherwise, setting $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ can force $\dot{V}_L(\mathbf{x})$ to be globally negative definite, and it follows from Lyapunov stability that trajectories converge to the origin. This contradicts the known property of $\mathbf{A}$, so there must be some $\mathbf{x} \neq \mathbf{0}$ belonging to $R_{[F_L \geq 0] \cup \{\mathbf{0}\}}$. This implies that $\mathbf{x}$ can be written as a linear combination of the eigenvectors of $\mathbf{M}$, and the maximum eigenvalue of $\mathbf{M}$ is non-negative. We denote this eigenvalue by $\lambda_{\mathbf{M}1}$ and note that $\mathbf{v}_{\mathbf{M}1} \in R_{[F_L \geq 0] \cup \{\mathbf{0}\}}$. ⊗

Now, we know by inspecting the expression of $\mathbf{G}_L(\mathbf{x})$ that $S_{\mathbf{G}_L = \mathbf{0}}$ is a subspace of dimension $n - rank(\mathbf{B})$. In addition, we recall from Lemma 1 that $S_{\mathbf{G}_L = \mathbf{0}} \subset R_{[F_L < 0] \cup \{\mathbf{0}\}}$. It follows that $R_{[F_L < 0] \cup \{\mathbf{0}\}}$ contains at least $n - rank(\mathbf{B})$ linearly independent vectors, along which $F_L(\mathbf{x}) < 0$. Since $V_\mathbf{M}$ is basis of $\Re^n$, it follows that $R_{[F_L < 0] \cup \{\mathbf{0}\}}$ contain at least $n - rank(\mathbf{B})$ eigenvectors of $\mathbf{M}$. Noticing this, it follows that at least $n - rank(\mathbf{B})$ eigenvalues of $\mathbf{M}$ are negative real. ⊗

Corollary 1: When $\mathbf{A}$ is unstable or marginally stable, $S_{F_L = 0}$ contains the origin and infinitely many other points.

Proof: It is clear that $F_L(\mathbf{0}) = 0$. Now, recall from Theorem 2 that when $\mathbf{A}$ is unstable or marginally stable, the maximum eigenvalue of $\mathbf{M}$ is non-negative, while at least $n - rank(\mathbf{B})$ eigenvalues of $\mathbf{M}$ are negative.

When the maximum eigenvalue of $\mathbf{M}$ is zero, it is clear that $F_L(\mathbf{x}) = 0$ at infinitely many points along $v_{\mathbf{M}1}$. When the maximum eigenvalue is positive, the fact that some eigenvalues of $\mathbf{M}$ are negative implies that $F_L(\mathbf{x})$ changes sign. Since $F_L(\mathbf{x})$ is a quadratic function, Corollary 1 follows. ⊗

From Theorem 2 and Corollary 1, we now know that whether $\mathbf{A}$ is stable or not, it is always true that 1) $\mathbf{0} \in S_{F_L = 0}$, and 2) $\lambda_{\mathbf{M}i} \in \Re \quad i = 1, \ldots, n$. When $\mathbf{A}$ is unstable or marginally stable, we know in addition that the maximum eigenvalue of $\mathbf{M}$ cannot be negative, while $\mathbf{M}$ has at least $n - rank(\mathbf{B})$ negative eigenvalues. Numerical computations show that the signs of the remaining eigenvalues, if any, are uncertain. When $\mathbf{A}$ is stable, it can be drawn from the proof of Theorem 2 that $\mathbf{M}$ also has at least $n - rank(\mathbf{B})$ negative real eigenvalues. However, the sign of the maximum eigenvalue of $\mathbf{M}$ can now be negative. Indeed, numerical examples show for this case that $\lambda_{\mathbf{M}1}$, the maximum eigenvalue of $\mathbf{M}$, can be negative, or zero, or positive. In all these cases, we arrange $\lambda_{\mathbf{M}1}$ as the maximum eigenvalues of $\mathbf{M}$, and $0 > \lambda_{\mathbf{M}g} \geq \ldots \geq \lambda_{\mathbf{M}n}$, where $g = rank(\mathbf{B}) + 1$.

Now, it is obvious that there are two possibilities for the geometry of $S_{F_L = 0}$. It contains either infinitely many points including the origin, or the origin only. The former happens when $\lambda_{\mathbf{M}1} \geq 0$ while the latter happens when $\lambda_{\mathbf{M}1} < 0$. On the other hand, the geometry of $S_{\mathbf{G}_L = \mathbf{0}}$ is unique, it is always a plane of dimension $n - rank(\mathbf{B})$. Corollary 2 states an important property of the intersection between these two zero surfaces:

Corollary 2: If $\mathbf{K}$ is such that $\bar{\mathbf{A}}$ is stable, then $S_{\mathbf{G}_L = \mathbf{0}} \cap S_{F_L = 0} = \{\mathbf{0}\}$

Proof: Knowing the existence of $S_{F_L = 0}$, the proof for Corollary 2 is immediate from Lemma 1. ⊗

When $\bar{\mathbf{A}}$ is stable, Corollary 2 states that the origin is the unique intersection point between the zero surfaces $S_{\mathbf{G}_L = \mathbf{0}}$ and $S_{F_L = 0}$. Later, we will discuss about possibilities for their relative orientations, and then examine how these can affect $\dot{V}_N(\mathbf{x}, t)$. In the next section, we examine properties of $S_{F_\Delta = 0}$ by referring to the properties of $S_{F_L = 0}$ given in this section.

5. Properties of Zero Surfaces in the Lyapunov Time Derivative along Trajectories of the Uncertain Systems with Multiple Inputs

In this section, we examine properties of the zero surfaces $S_{F_\Delta = 0}$ associated with $\dot{V}_N(\mathbf{x}, t)$. Then we use these and the known properties of $S_{F_L = 0}$ and $S_{\mathbf{G}_L = \mathbf{0}}$ to draw in the next section a critical situation in which no class-gamma theorem can guarantee stability of the uncertain systems. To begin with, we recall that $F_\Delta(\mathbf{x}, t) = F_L(\mathbf{x}) + F_\Omega(\mathbf{x}, t)$. From this, we elect to examine geometry of the zero surface $S_{F_\Delta = 0}$ by referring to known properties of the zero surface $S_{F_L = 0}$. An important property of $S_{F_\Delta = 0}$ is given in the following Lemma 2:

Lemma 2: If $S_{F_L = 0}$ contains infinitely many points and $F_\Omega(\mathbf{x}, t)$ is sufficiently small, then $\mathbf{x}_{F_\Delta = 0} \in S_{F_\Delta = 0}$ is in a correspondingly small neighborhood about $\mathbf{x}_{F_L = 0} \in S_{F_L = 0}$, or is vanishing.

Proof: Let $\mathbf{x}_{F_L = 0} \in S_{F_L = 0}$, and define the region $U = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_{F_L = 0}\| \leq \delta\}$. When $\lambda_{\mathbf{M}1} > 0$, we have that $\sup_{\mathbf{x} \in U} F_L(\mathbf{x}) > 0$ and $\inf_{\mathbf{x} \in U} F_L(\mathbf{x}) < 0$. Now, define $\gamma = \min(|\sup_{\mathbf{x} \in U} F_L(\mathbf{x})|, |\mathrm{int}_{\mathbf{x} \in U} F_L(\mathbf{x})|)$, where $|a|$ denotes the absolute value of $a \in \Re$. When $F_\Omega(\mathbf{x}, t)$ is sufficiently small such that $\gamma > \max(|\sup_{\mathbf{x} \in U} F_\Omega(\mathbf{x}, t)|, |\inf_{\mathbf{x} \in U} F_\Omega(\mathbf{x}, t)|)$, we have $\sup_{\mathbf{x} \in U} F_\Delta(\mathbf{x}, t) > 0$ and $\inf_{\mathbf{x} \in U} F_\Delta(\mathbf{x}, t) < 0$ because $F_\Delta(\mathbf{x}, t) = F_L(\mathbf{x}) + F_\Omega(\mathbf{x}, t)$. Using the intermediate-value theorem, it follows that $F_\Delta(\mathbf{x}, t) = 0$ at some $\mathbf{x} \in U$. Repeating this for all $\mathbf{x}_{F_L = 0} \in S_{F_L = 0}$ shows that $S_{F_\Delta = 0}$ is in a neighborhood about $S_{F_L = 0}$, which is defined by the union of all the corresponding U. It is clear that if $F_\Omega(\mathbf{x}, t)$ is smaller than this, then we can take a smaller γ in the above process. This implies that $S_{F_\Delta = 0}$ is in the correspondingly small neighborhood about $S_{F_L = 0}$. When $\lambda_{\mathbf{M}1} = 0$, similar arguments can be employed to support the remaining statement. ⊗

For the case in which $S_{F_L = 0} = \{\mathbf{0}\}$, we see by inspecting the expression for $F_\Delta(\mathbf{x}, t)$ that $S_{F_\Delta = 0} = \{\mathbf{0}\}$ when $F_\Omega(\mathbf{x}, t)$ is negative or zero. So we have in these cases that the two zero surfaces have the same geometry. However, this is not true as $F_\Omega(\mathbf{x}, t)$ is increasing from zero. We now examine in Lemma 3 possible geometrical differences between these two zero surfaces in the latter situation. In Lemma 3, we let B denotes $\{\mathbf{x} : \|\mathbf{x}\| = c, c \in \Re^+\}$, $\mathbf{x}_B$ denotes a point in B, $\mathbf{x}_B^*$ denotes $\mathbf{x}_B$ that is located along $\mathbf{v}_{\mathbf{M}1}$, and $\mathbf{x}_B$ is denoted in the principal basis of $\mathbf{M}$ by $\mathbf{z}_B = [z_{B1} \ \ldots \ z_{Bn}]^T$.

Lemma 3: If $S_{F_L = 0} = \{\mathbf{0}\}$ and $F_\Omega(\mathbf{x}_B, t)$ increases indefinitely from zero, then $S_{F_\Delta = 0}$ contains points in B as $F_\Omega(\mathbf{x}_B, t) = -F_L(\mathbf{x}_B)$. In this situation, the first point in B that is contained in $S_{F_\Delta = 0}$ is $\mathbf{x}_B = \mathbf{x}_B^*$.

Proof: When $S_{F_L = 0} = \{\mathbf{0}\}$, we have that $F_L(\mathbf{x}_B) < 0 \ \forall \mathbf{x}_B$. If we have in addition that $F_\Delta(\mathbf{x}, t) = 0$, then it follows from the definition of $F_\Delta(\mathbf{x}, t)$ that $F_\Omega(\mathbf{x}_B, t) = -F_L(\mathbf{x}_B) > 0$. Knowing these, we can prove Lemma 4 by showing that $\max(F_L(\mathbf{x}_B)) = F_L(\mathbf{x}_B^*)$. Since $\mathbf{M}$ is real symmetric, this is immediate from Rayleigh's principle [11].

We now summarize our findings from Lemma 2 and 3. As uncertainties increase from zero, $|F_\Omega(\mathbf{x}, t)|$ can increase from zero to certain values, and this can cause geometrical differences between the zero surfaces $S_{F_\Delta = 0}$ and $S_{F_L = 0}$. When $S_{F_L = 0}$ contains infinitely many point and $|F_\Omega(\mathbf{x}, t)|$ is small, $S_{F_\Delta = 0}$ is contained in small neighborhood about $S_{F_L = 0}$, or is vanishing. When $S_{F_L = 0} = \{\mathbf{0}\}$ and $F_\Omega(\mathbf{x}, t)$ is non-positive, geometry of $S_{F_\Delta = 0}$ is the same as that of $S_{F_L = 0}$. As $F_\Omega(\mathbf{x}, t)$

increases to certain positive values, $S_{F_\Delta=0}$ will appear first in small neighborhood about the axis along $\mathbf{v_{M1}}$.

Despite of the fact that $S_{F_L=0}$ intersects $S_{\mathbf{G}_L=\mathbf{0}}$ only at the origin, it appears in many numerical examples in which geometrical differences between $S_{F_L=0}$ and $S_{F_\Delta=0}$ are small that $S_{F_\Delta=0}$ can intersect $S_{\mathbf{G}_L=\mathbf{0}}$ at nonzero points. In the next section, we show that these nonzero intersection points are highly undesirable when applying a class-gamma theorem. Then, we use the known properties of these zero surfaces to show how the occurrence of these nonzero intersection points may be avoided when geometrical differences between $S_{F_L=0}$ and $S_{F_\Delta=0}$ are small.

6. A Necessary Condition for all Class-Gamma Theorems

If a pair $(\mathbf{K},\mathbf{Q})$ satisfies a class-gamma theorem such that $a_\gamma < 0$, then it is necessary that such $(\mathbf{K},\mathbf{Q})$ forces $\dot{V}_N(\mathbf{x},t)$ to be uniformly globally negative definite. In Lemma 5, we examine the critical situation in which this cannot happen under all $(\mathbf{K},\mathbf{Q})$.

Lemma 4: For $\dot{V}_N(\mathbf{x},t)$ to be uniformly globally negative definite, it is necessary that $S_{F_\Delta=0}$ intersects $S_{\mathbf{G}_L=\mathbf{0}}$ only at the origin.

Proof: We assume that these two zero surfaces intersect at a nonzero point $\mathbf{x}^\otimes$. Now, because we have at $\mathbf{x}^\otimes$ that $F_\Delta(\mathbf{x}^\otimes,t)=0$ and $\mathbf{G}_L(\mathbf{x}^\otimes)=\mathbf{0}$, it is obvious that $\dot{V}_N(\mathbf{x}^\otimes,t)=0$ no matter what $\mathbf{u}(\mathbf{x}^\otimes)$ is. This condition implies that $\dot{V}_N(\mathbf{x},t)$ is not uniformly globally negative definite. ⊗

It is clear that uncertainties are the cause of geometrical differences between $S_{F_L=0}$ and $S_{F_\Delta=0}$ and the undesirable possibility that $S_{F_\Delta=0}$ intersects $S_{\mathbf{G}_L=\mathbf{0}}$ at a nonzero point. When uncertainties are large, it may be that $S_{F_\Delta=0}$ considerably deviates from $S_{F_L=0}$. In this case, it may be that this undesirable possibility cannot be avoided, and no class-gamma theorem can guarantee stability of the uncertain systems. The question is whether we can avoid this when uncertainties and geometrical differences between these two zero surfaces are reasonably small.

For the case in which $S_{F_L=0}$ contains infinitely many points, it appears in many numerical examples that $S_{\mathbf{G}_L=\mathbf{0}}$ can be very close to a particular portion of $S_{F_L=0}$ such that small geometrical differences between $S_{F_\Delta=0}$ and $S_{F_L=0}$ in that portion cause nonzero intersection points between $S_{F_\Delta=0}$ and $S_{\mathbf{G}_L=\mathbf{0}}$. For the case in which $S_{F_L=0}=\{\mathbf{0}\}$, $S_{F_\Delta=0}$ can intersect $S_{\mathbf{G}_L=\mathbf{0}}$ at a nonzero point as well if $S_{\mathbf{G}_L=\mathbf{0}}$ is very close to a particular portion of the axis along $\mathbf{v_{M1}}$. Noticing these, we want to find $(\mathbf{K},\mathbf{Q})$ such that $S_{\mathbf{G}_L=\mathbf{0}}$ is not close to a particular portion of $S_{F_L=0}$, nor a particular portion of the axis along $\mathbf{v_{M1}}$. We conclude this as:

Condition of Symmetry [10]*:* When $S_{F_L=0}$ contains infinitely many points, the zero surface $S_{\mathbf{G}_L=\mathbf{0}}$ is symmetric about the zero surface $S_{F_L=0}$. When $S_{F_L=0}=\{\mathbf{0}\}$, the zero surface $S_{\mathbf{G}_L=\mathbf{0}}$ is symmetric about the axis along the eigenvector $\mathbf{v_{M1}}$.

We emphasize that the relative orientation of the zero surfaces specified by the condition of symmetry may not be the most suitable for a specific set of uncertainty specifications and a specific class-gamma theorem. However, the condition offers a "safe" relative orientation for these surfaces, which is reasonable under all uncertainties and all class-gamma theorems. It is this versatility that motivates the use of the condition of symmetry in our procedure for extending the uses of class-gamma theorems over robust controller design. In the next section, we show that the condition of symmetry can be satisfied by a special set of $(\mathbf{K},\mathbf{Q})$.

7. Obtaining (K, Q) to Satisfy the Condition of Symmetry

According to the known properties of zero surfaces, the condition of symmetry is satisfied if we have simultaneously that:

S1) When $S_{F_L=0}=\{\mathbf{0}\}$, $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ is symmetric about the axis along the eigenvector $\mathbf{v_{M1}}$.

S2) When $S_{F_L=0}$ contains infinitely many points, $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ is symmetric about $S_{F_L=0}$.

S3) $S_{\mathbf{G}_L=\mathbf{0}}=P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$

S4) Lemma 1 is satisfied.

To show that these requirements can be satisfied simultaneously, we introduce several theorems and lemmas in this section. To begin with, recall that the n eigenvectors of $\mathbf{M}$ are orthonormal. Knowing this, it is trivial to show that the axis along $\mathbf{v_{M1}}$ is orthogonal to the plane $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$. Clearly, this implies that the plane $P_{\{\mathbf{v_{M2}},\ldots,\mathbf{v_{M}}_n\}}$ is symmetric about the axis along $\mathbf{v_{M1}}$, so we skip the proof for this and for S1). Next, we show in Lemma 5 that S2) is true:

Lemma 5: When $S_{F_L=0}$ contains infinitely many points, $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ is symmetric about the zero surface $S_{F_L=0}$.

Proof: In this case, $\lambda_{\mathbf{M1}}$ is non-negative, the signs of $\lambda_{\mathbf{M2}},\ldots,\lambda_{\mathbf{M}g-1}$ are uncertain, and $\lambda_{\mathbf{M}g},\ldots,\lambda_{\mathbf{M}n}<0$. Now, we consider the case in which $\lambda_{\mathbf{M1}}>0$. Using the linear transformation $\mathbf{x}=\mathbf{T_M z}$ and the transformation matrix $\mathbf{T_M}=[\mathbf{v_{M1}}\mid\ \ldots\ \mid\mathbf{v_{M}}_n]$, we obtain the expression for $F_L(\mathbf{x})$:

$$F_L(\mathbf{x})=\mathbf{x}^T\mathbf{M}\mathbf{x}=\mathbf{z}^T[\mathbf{T_M}^T\mathbf{M}\mathbf{T_M}]\mathbf{z}\equiv\mathbf{z}^T\mathbf{\Lambda_M}\mathbf{z}=F_L(\mathbf{z}) \qquad (18)$$

Direct expansion yields:

$$F_L(\mathbf{z})=\lambda_{\mathbf{M1}}z_1^2+\ldots+\lambda_{\mathbf{M}g-1}z_{g-1}^2+\lambda_{\mathbf{M}g}z_g^2+\ldots+\lambda_{\mathbf{M}n}z_n^2 \qquad (19)$$

We denote in the principal basis of $\mathbf{M}$ a point in $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ by $\mathbf{z}=[0\ \ldots\ 0\ z_g\ \ldots\ z_n]^T\equiv\mathbf{z}_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$. We see that $F_L(\mathbf{z}_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}})<0$, except at the origin where $F_L(\mathbf{0})=0$. The expansion implies the existence of a sufficiently large value for z_1 such that $F_L(\mathbf{z})=0$ at $\mathbf{z}=[z_1\ \ldots\ z_{g-1}\ z_g\ \ldots\ z_n]^T\equiv\mathbf{z}_{F_L=0,1}$. The vector joining $\mathbf{z}_{F_L=0,1}$ and $\mathbf{z}_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ is:

$$[z_1\ \ldots\ z_{g-1}\ 0\ \ldots\ 0]^T\equiv\mathbf{z}_{\Delta1}$$

Now, notice that $F_{LT}(\mathbf{z})=0$ when

$$\mathbf{z}=[-z_1\ \ldots\ -z_{g-1}\ z_g\ \ldots\ z_n]^T\equiv\mathbf{z}_{F_L=0,2}$$

The vector joining $\mathbf{z}_{F_L=0,2}$ and $\mathbf{z}_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ is:

$$[-z_1\ \ldots\ -z_{g-1}\ 0\ \ldots\ 0]^T\equiv\mathbf{z}_{\Delta2}$$

Since the lengths of $\mathbf{z}_{\Delta1}$ and $\mathbf{z}_{\Delta2}$ are the same, we assert the symmetry of $P_{\{\mathbf{v_{M}}_g,\ldots,\mathbf{v_{M}}_n\}}$ about $S_{F_L=0}$. When the maximum eigenvalue of $\mathbf{M}$ is zero with multiplicity q, the expansion of $F_L(\mathbf{z})$ implies that $S_{F_L=0}$ is spanned the corresponding set of q orthonormal eigenvectors other than $\mathbf{v}_{\mathbf{M}_g},\ldots,\mathbf{v}_{\mathbf{M}_n}$. The symmetry is immediate from this observation. ⊗

Now, we have shown that S1) and S2) are true without imposing constraints on $(\mathbf{K},\mathbf{Q})$. In the followings, the objective is to show that S3), and S4) can be satisfied simultaneously when $(\mathbf{K},\mathbf{Q})$ is obtained in a special fashion. We now define the matrix:

$$\mathbf{N}\equiv(1/2)[[\mathbf{PB}]\mathbf{K}+\mathbf{K}^T[\mathbf{PB}]^T] \qquad (20)$$

where $\mathbf{N}$ is drawn from the Lyapunov equation after substituting $\mathbf{A}-\mathbf{BK}$ for $\bar{\mathbf{A}}$. The first step is to show some relationships between the eigenvectors of $\mathbf{M}$ and $\mathbf{N}$ under special choices for $\mathbf{Q}$. These are given in Lemma 6:

Lemma 6: If $\mathbf{Q}=c\mathbf{I}$, $c\in\Re^+$, the sets of eigenvectors of $\mathbf{N}$ and of $\mathbf{M}$ are the same. In addition, the eigenvector $\mathbf{v_{M1}}$ is an eigenvector corresponding to the maximum eigenvalue of $\mathbf{N}$.

Proof: Substituting $[\mathbf{A}-\mathbf{BK}]$ for $\bar{\mathbf{A}}$, and $c\mathbf{I}$, $c\in\Re^+$ for $\mathbf{Q}$ in the Lyapunov equation produces:

$$-\mathbf{Q}=-c\mathbf{I}=(1/2)[\mathbf{PA}+\mathbf{A}^T\mathbf{P}]-(1/2)[[\mathbf{PB}]\mathbf{K}+\mathbf{K}^T[\mathbf{PB}]^T]\equiv\mathbf{M}-\mathbf{N} \qquad (21)$$

We premultiply and postmultiply every term in Eq. (21) by $\mathbf{T_M}^T$ and $\mathbf{T_M}$ respectively to obtain:

$$-c\mathbf{I} = \mathbf{\Lambda_M} - \mathbf{T_M^T N T_M} \quad (22)$$

Since $\mathbf{I}$ and $\mathbf{\Lambda_M}$ are diagonal, $\mathbf{T_M^T N T_M} \equiv \mathbf{\Lambda_N}$ is diagonal, and thus implying that $\mathbf{N}$ can be diagonalized by using $\mathbf{v}_{\mathbf{M}i}$, $i = 1, ..., n$. Accordingly, $\mathbf{N}$ and $\mathbf{M}$ have the same set of eigenvectors, and $\mathbf{\Lambda_N} \equiv \text{diag}[\lambda_{\mathbf{N}1} \ \dots \ \lambda_{\mathbf{N}n}]$. Now, because $\lambda_{\mathbf{M}1}$ is the maximum eigenvalue of $\mathbf{M}$, it follows from Eq. (22) that $\lambda_{\mathbf{N}1}$ is the maximum eigenvalue of $\mathbf{N}$. Since $\mathbf{M}$ and $\mathbf{N}$ share the same set of eigenvectors, $\mathbf{v}_{\mathbf{M}1}$ is an eigenvector of $\mathbf{N}$ corresponding to $\lambda_{\mathbf{N}1}$. ⊗

Matrix theory was employed in [10] to arrive at the preliminary conclusion that S3) and S4) are satisfied simultaneously when $\mathbf{Q} = c\mathbf{I}$ and $\mathbf{K} = \rho\mathbf{B}^T\mathbf{P}$. For the present case of multiple input systems, we reverse the argument to shorten the proof. This is accomplished by showing that under the preliminary choices of $\mathbf{Q} = c\mathbf{I}$ and $\mathbf{K} = \rho\mathbf{B}^T\mathbf{P}$, S3) and S4) are satisfied simultaneously.

Theorem 3: If $\mathbf{Q} = c\mathbf{I}$ and $\mathbf{K} = \rho\mathbf{B}^T\mathbf{P}$, where $c, \rho \in \Re^+$, then $S_{\mathbf{G}_L=0} = P_{\{\mathbf{v}_{\mathbf{M}g}, \dots, \mathbf{v}_{\mathbf{M}n}\}}$ and Lemma 1 is satisfied.

Proof: With $\mathbf{u}(\mathbf{x}) = -\mathbf{K}\mathbf{x} = -\rho\mathbf{B}^T\mathbf{P}\mathbf{x}$, we obtain:

$$\mathbf{G}_L(\mathbf{x})\mathbf{u}(\mathbf{x}) = -\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} = -\rho[\mathbf{x}^T\mathbf{P}\mathbf{B}][\mathbf{x}^T\mathbf{P}\mathbf{B}]^T = -\mathbf{G}_L(\mathbf{x})\mathbf{G}_L^T(\mathbf{x}) \le 0 \quad (23)$$

where $\bar{\mathbf{N}} \equiv \rho[\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P}] = \bar{\mathbf{N}}^T$, and $\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} \ge 0$. We know that:

1) The real symmetric matrix $\bar{\mathbf{N}}$ has a set of n orthonormal eigenvectors $V_{\bar{\mathbf{N}}} \equiv \{\mathbf{v}_{\bar{\mathbf{N}}1}, \dots, \mathbf{v}_{\bar{\mathbf{N}}n}\}$ spanning $\Re^n$.
2) For $i = 1, \dots n$, $\lambda_{\bar{\mathbf{N}}i} \ge 0$ because $\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} \ge 0$ and $\bar{\mathbf{N}} = \bar{\mathbf{N}}^T$.

Now, Eq. (23) implies that:

$$S_{\mathbf{G}_L=\mathbf{0}} = \{\mathbf{x} \,|\, \mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} = 0\} \quad (24)$$

Because $\dim(S_{\mathbf{G}_L=\mathbf{0}}) = n - rank(\mathbf{B})$ and because the n eigenvectors of $\bar{\mathbf{N}}$ are orthonormal, we know that a basis of $S_{\mathbf{G}_L=\mathbf{0}}$ is a set of $n - rank(\mathbf{B})$ orthonormal eigenvectors of $\bar{\mathbf{N}}$. For convenience, we arrange the vectors in this basis as $\{\mathbf{v}_{\bar{\mathbf{N}}g}, \dots, \mathbf{v}_{\bar{\mathbf{N}}n}\}$ and denote the set of the corresponding eigenvalues by $\{\lambda_{\bar{\mathbf{N}}g}, \dots, \lambda_{\bar{\mathbf{N}}n}\}$. Eq. (24) implies that these are all the zero eigenvalues. Indeed, if $\bar{\mathbf{N}}$ has other zero eigenvalues then $\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} = 0$ along the corresponding orthonormal eigenvectors, implying that $\dim(S_{\mathbf{G}_L=\mathbf{0}}) > n - rank(\mathbf{B})$. This is a contradiction, and thus $\bar{\mathbf{N}}$ has exactly $n - rank(\mathbf{B})$ zero eigenvalues. The remaining $rank(\mathbf{B})$ eigenvalues of $\bar{\mathbf{N}}$ are positive because $\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} \ge 0$.

Now, we have in the orthonormal basis of $\mathbf{M}$ and $\bar{\mathbf{N}}$ that:

$$-c\mathbf{I} = \mathbf{\Lambda_M} - \mathbf{\Lambda}_{\bar{\mathbf{N}}} \quad (25)$$

where $\mathbf{\Lambda}_{\bar{\mathbf{N}}} = \text{diag}[\lambda_{\bar{\mathbf{N}}1} \ \dots \ \lambda_{\bar{\mathbf{N}}n}]$. We see that $\lambda_{\mathbf{M}i} = -c < 0$ because $\lambda_{\bar{\mathbf{N}}i} = 0$, $i = g, \dots, n$, and $\mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} = 0$ on the space spanned by the corresponding $\mathbf{v}_{\mathbf{M}i}$. Since $\{\mathbf{x} \,|\, \mathbf{x}^T\bar{\mathbf{N}}\mathbf{x} = 0\} = S_{\mathbf{G}_L=0}$, it follows that $\mathbf{v}_{\mathbf{M}i}$ spans $S_{\mathbf{G}_L=0}$ such that $\lambda_{\mathbf{M}i} < 0$. Accordingly, S3) is satisfied.

The imposed choices of $\mathbf{Q}$, and $\mathbf{K}$ produce the Riccati equation:

$$-2c\mathbf{I} = \mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - 2\rho\mathbf{P}\mathbf{B}\mathbf{B}^T\mathbf{P} \quad (26)$$

Existence and uniqueness of the solution $\mathbf{P}$ of Eq. (26) is guaranteed, provided that [$\mathbf{A}$, $\mathbf{B}$] is controllable or stabilizable, and the gain matrix $\mathbf{K} = \rho\mathbf{B}^T\mathbf{P}\mathbf{x}$ is a stabilizing solution [12]. ⊗

In Theorem 3, we restrict that $\mathbf{Q} = c\mathbf{I}$ when obtaining ($\mathbf{K}$, $\mathbf{Q}$) satisfying the condition of symmetry. In the following Theorem 4, we present our final result without this restriction.

Theorem 4: If $\mathbf{P}$ is the solution of the Riccati equation (26), then setting $\mathbf{K} = \eta\rho\mathbf{B}^T\mathbf{P}\,|_{\eta\ge1}$ produces $(\mathbf{K},\mathbf{Q})$ satisfying the condition of symmetry, and $\mathbf{Q}$ need not be $c\mathbf{I}$. The nominal linear model is guaranteed to be stable.

Proof: With $\mathbf{P}$ obtained from the Riccati equation, substituting $\eta\rho[\mathbf{P}\mathbf{B}]$ for $\mathbf{K}^T$ in Eq. (21) yields:

$$\mathbf{Q} = -\mathbf{M} + \mathbf{N} + (\eta - 1)\mathbf{N} = c\mathbf{I} + (\eta - 1)\mathbf{N} \quad (27)$$

Since $\eta \ge 1$, $\mathbf{N} = \mathbf{N}^T$ and $\mathbf{N} \ge 0$, Eq. (27) implies that $\mathbf{Q}$ is symmetric positive definite and $\mathbf{Q}$ need not be $c\mathbf{I}$. By Lyapunov stability, this guarantees stability of the nominal linear model. Substituting $\eta\rho[\mathbf{P}\mathbf{B}]$ for $\mathbf{K}^T$ in Eq. (21) does not alter the condition of symmetry, because the eigenvectors and the eigenvalues of $\mathbf{M}$, and $S_{\mathbf{G}_L=\mathbf{0}}$ are the same. ⊗

Using Theorem 4, we now can generate the pairs $(\mathbf{K},\mathbf{Q})$ that satisfy the condition of symmetry without restricting that $\mathbf{Q} = c\mathbf{I}$. These are potential candidates for solutions of all class-gamma theorems, and for our robust controllers.

8. Using a Class-Gamma RSA Theorem for Robust Controller Design

To obtain a robust controller, we begin by selecting a class-gamma theorem to match the available uncertainty specifications. Then, we determine if a candidate ($\mathbf{K}$, $\mathbf{Q}$) with the property of symmetry, or a pair of matrices with correct dimensions located nearby, is a solution for the selected class-gamma theorem. As in [10], we have for the present case the advantageous property that the candidates are generated from the two scalar parameters ρ and η. Accordingly, we simply plot a_γ versus these two parameters in 3D and select from the plot the coordinate at which $a_\gamma < 0$ to find the solution ($\mathbf{K}$, $\mathbf{Q}$). Simple univariate numerical search [13] can be employed to find solutions located nearby the candidates when starting at points corresponding to small values of a_γ obtained from the plot.

9. Example

Consider the problem of designing a robust linear controller for a helicopter [14] about an operating point. For a range of wind speed, the helicopter dynamics is represented by:

$$\dot{\mathbf{x}} = [\mathbf{A} + h_1(\mathbf{x},t)\mathbf{E}_1 + h_2(\mathbf{x},t)\mathbf{E}_2]\mathbf{x} + [\mathbf{B} + h_3(\mathbf{x},t)\mathbf{E}_3]\mathbf{u}$$

where $\mathbf{A} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.707 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0.4422 & 0.1761 \\ 0.0447 & -7.5922 \\ -5.52 & 4.99 \\ 0 & 0 \end{bmatrix}$,

$\mathbf{E}_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix}$, $\mathbf{E}_2 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$, $\mathbf{E}_3 = \begin{bmatrix} 0&0 \\ 1&0 \\ 0&0 \\ 0&0 \end{bmatrix}$, $h_1(\mathbf{x},t) \in [-0.2192, 0.2192]$, $h_2(\mathbf{x},t) \in [-1.2031, 1.2031]$, and $h_3(\mathbf{x},t) \in [-2.0673, 2.0673]$. The objective is to find $\mathbf{K}$ for $\mathbf{u} = -\mathbf{K}\mathbf{x}$ to stabilize the system for all possible uncertain functions $h_i(\mathbf{x},t)$, i = 1, 2, and 3. The nominal linear model is unstable because two of the eigenvalues $\mathbf{A}$ are in the RHP. For the specified uncertainty specifications, a stabilizing solution for this problem can be found in [14].

We demonstrate the effectiveness of the proposed technique by showing that our linear controller can stabilize the system when the uncertainties increases by 20%, or $h_1(\mathbf{x},t) \in 1.2[-0.2192 \ ,0.2192]$, $h_2(\mathbf{x},t) \in 1.2[-1.2031, \ 1.2031]$, and $h_3(\mathbf{x},t) \in 1.2[-2.0673, \ 2.0673]$. Our controller design process starts by selecting an appropriate class-gamma RSA theorem, then employ Theorem 3 and 4 to construct a 3D plot of a_γ versus ρ, and η. Using $\lambda_{\mathbf{Z}1}$ in Theorem 1 as a_γ produces the 3D plot in Fig. 1. Data shows that there are infinitely many points (ρ, η) at which $\lambda_{\mathbf{Z}1} < 0$. We simply select $(\rho, \eta) = (0.06, 1.6)$, which corresponds to $\lambda_{\mathbf{Z}1} = -0.0052$. The corresponding state-feedback gain matrix is:

$$\mathbf{K} = \begin{bmatrix} 0.3448 & -0.0113 & -0.3888 & -0.5856 \\ -0.0401 & -0.2724 & 0.1715 & 0.4940 \end{bmatrix}$$

The four eigenvalues of $\bar{\mathbf{A}}$ are located at s = $-0.8597 \pm \text{j}1.0525$, - 4.8231, and - 0.3922 in the complex plane.

Although not required theoretically, we provide simulations in Fig. 2 for completeness. In these simulations, the assumed uncertain functions are $h_1(\mathbf{x},t) = 1.2(0.2192)\sin(x_1x_2t)$ $h_2(\mathbf{x},t) = 1.2(1.2031)\sin(x_3x_4)$ and $h_3(\mathbf{x},t) = 1.2(2.0673)\cos(x_1x_3)$. From two initial conditions $\mathbf{x}_{0,1}$ and $\mathbf{x}_{0,2}$, simulations confirm that our linear control can force the trajectories to converge to the origin.

The above enlarged allowable uncertainty bounds are not the least conservative bounds we can obtain. By using the simple univariate search [13], we can find stabilizing controls for up to 72% increases in uncertainties when the search starts from initial condition corresponding to small values of $a_\gamma = \lambda_{\mathbf{Z}1}$ obtained from the 3D plot. We start the univariate search from (**K**, **Q**) corresponding to $(\rho, \eta) = (0.2, 1.275)$, and $\lambda_{\mathbf{Z}1} = 1.3546$. The search took approximately 1 minute on a 400 MHz PC, and yielded the linear state-feedback gain matrix:

$$\mathbf{K} = \begin{bmatrix} 0.4604 & -0.0310 & -0.9121 & -1.1086 \\ -0.0378 & -0.5740 & 0.1841 & 0.8699 \end{bmatrix}$$

This corresponds to $\lambda_{\mathbf{Z}1} = -1.6585 \times 10^{-3}$, and places the eigenvalues of $\bar{\mathbf{A}}$ at $s = -1.5465 \pm j0.6234$, -0.3304, and -8.7442 in the complex plane. Simulations in Fig. 3 confirm stability of the resulting control system. It appears in our investigation that the search fails to find a solution if the initial **K** is simply selected to stabilize $\bar{\mathbf{A}}$, and the initial **Q** is simply selected to be symmetric positive definite.

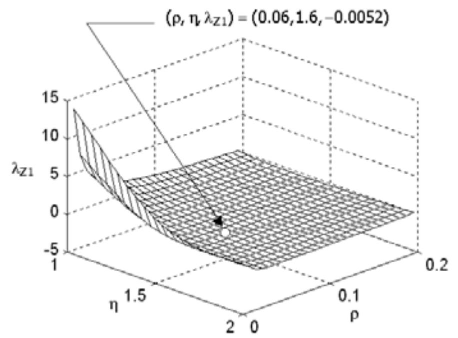

Fig. 1 3D Plot of $\lambda_{\mathbf{Z}1}$ from Theorem 1 versus ρ, and η for 12 % Increases in Uncertainties

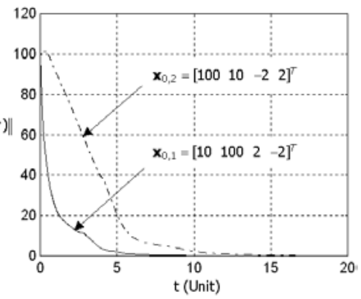

Fig. 2 Simulation Results when Uncertainties Increase 12%

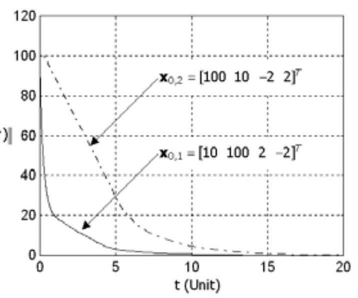

Fig. 2 Simulation Results when Uncertainties Increase 72%

10. Conclusion

For a linear control system, a class-gamma robust stability analysis (RSA) theorem guarantees that the system remains stable under nonlinear time-varying uncertainties by showing that the time-derivative of a quadratic Lyapunov function along trajectories of the uncertain system is uniformly globally negative definite if a certain inequality $a_\gamma < 0$ is satisfied. Many of these theorems have been proposed through decades, and they now available for various classes of uncertainties. Motivated by this fact, applications of these theorems were recently extended to cover robust control of the uncertain linear systems with single input. This paper extends this result further to cover robust control design of the uncertain linear systems with multiple inputs.

In this paper, it was shown that a certain relative orientation of zero surfaces associated with the Lyapunov time derivative along trajectories of the nominal linear model is particularly useful when requiring that the class-gamma theorem of interest be satisfied. Two theorems were proposed to generate controllers that guarantee this relative orientation. It turns out that these theorems require only two scalar parameters to generate such controllers. This is the most advantageous property as it allows us to plot a_γ versus the two parameters in 3D, and simply select from the plot a coordinate at which $a_\gamma < 0$ to compute the corresponding stabilizing solution. For increased uncertainties, simple numerical searchs can be used to find stabilizing solutions when the initial conditions correspond to small values of a_γ in the 3D plot. Examples show that our controllers can guarantee stability of multiple-input linear systems such that the resulting allowable uncertainty bounds are satisfactorily large.

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