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The Method of Fundamental Solutions for Seepage Problem

Khomgris Chaiyo[1*] and Somchart Chantasiriwan[2**]

[1] Spansion (Thailand) Limited
Pakkred, Nonthaburi 11120, Thailand
Tel: 0-296-1180, Fax: 0-2962-2046, [*]E-mail: khomgris.chaiyo@spansion.com
[2] Department of Mechanical Engineering, Faculty of Engineering, Thammasat University
Khlong Luang, Pathum Thani 12121, Thailand
Tel: 0-2564-6304, Fax: 0-2564-6303, [**]E-mail: somchart@engr.tu.ac.th

Abstract

The problem of seepage flow through a dam is moving boundary problem that is more conveniently solved by a meshless method than a mesh-based method such as finite element method (FEM) and finite difference method (FDM). This paper presents method of fundamental solutions, which is one kind of meshless methods, to solve a dam problem using the fundamental solution to the Laplace's equation. Solutions on free surface are determined by iteration and cubic spline interpolation. The numerical solutions then are compared with the boundary element method (BEM), FDM and FEM to display the performance of present method.

Keywords: method of fundamental solutions, seepage, moving boundary

1. Introduction

The two-dimensional steady state saturated isotropic seepage flow with moving boundary is described by the Laplace equation necessary boundary conditions. Conventional methods used to solve such problem include FDM, and FEM. These methods are all mesh-dependent methods because they require boundary-fitted mesh generation. Alternative methods include BEM and MFS. Both methods do not require boundary-fitted mesh, which results in considerable simplification of the preprocessing step. MFS has additional advantages over BEM in that it requires only boundary node placement instead of boundary mesh generation, and it does not require evaluation of near singular integrals [1]. The basic idea of MFS is to approximate the solution by forming a linear combination of known fundamental solutions with sources located outside the problem domain.

In order to study seepage problem, accurately defining the position of free surface is very important and necessary. In the past, many researchers utilized several methods to determine the location of free surface such as Aitchison [2], and Westbrook [3] used FDM and FEM, to solve the position of the moving boundary, respectively. The conventional BEM was then used to study the seepage flow through the porous media by Liggett and Liu [4], and also BEM using hypersingular equations was proposed by Chen et al. [5].

In this paper, free surface is regarded as a moving boundary with the over-specified boundary conditions, and MFS is used to find the location of free surface. The numerical results of present method are also compared with FDM, FEM, and BEM solutions.

2. The Seepage Problem

The governing equation of two-dimensional steady-state isotropic seepage flow through a dam can be described by the Laplace equation as

$$\nabla^2\varphi = 0 \qquad (1)$$

where φ is the velocity potential. Consider a dam is also the free surface seepage problem shown as Figure 1. The piezometric head can be written as

$$\varphi = y + \frac{p}{\gamma} \qquad (2)$$

where y is the position, p is the pressure, and γ is the specific gravity of fluid [6]. Therefore, the boundary conditions are presented in Figure 1.

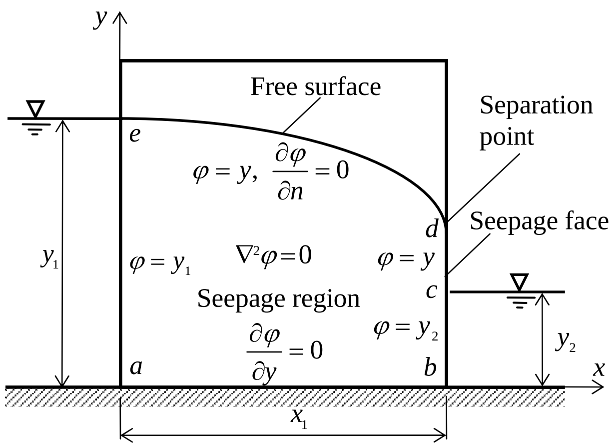

Figure 1. Flow through a 2D rectangular dam

For moving surface (d-e), the pressure head equal to zero, thus velocity potential function can be expressed as

$$\varphi = y \tag{3}$$

The moving boundary is the interface between saturated region and dry region. Boundary conditions at the moving surface are over-specified. In the following section, this surface will be determined by using MFS with the initial guess of moving boundary.

3. The Method of Fundamental Solutions

For basic idea of MFS is to express φ as linear combination of fundamental solutions [7]. Consider Figure 1, let D is seepage region that be a bounded, simply connected domain in R^2 with boundary S. On boundary b-c, c-d, and a-e are Dirichet boundary, and boundary a-b is Neumann boundary. Boundary d-e is combined Dirichet and Neumann boundary, or is called as Robin boundary. For these boundaries can generally expressed as

$$\varphi = f(x, y) \qquad \text{for } (x, y) \text{ on } S_1 \tag{4}$$

$$n_x \frac{\partial \varphi}{\partial x} + n_y \frac{\partial \varphi}{\partial y} = g(x, y) \qquad \text{for } (x, y) \text{ on } S_2 \tag{5}$$

where direction cosine n_x and n_y are x-, and y-components, respectively, of the outward normal unit vector. The fundamental solution satisfies the solution of Laplace's equation as

$$G(P_i, Q_j) = \frac{1}{2\pi} \log r_{ij} \qquad (P_i \in D, Q_j \in \hat{S}) \tag{6}$$

where

$$r_{ij} = \sqrt{(x_i - \xi_j)^2 + (y_i - \eta_j)^2} \tag{7}$$

is an Euclidian distance between two collocation points, and (ξ_j, η_j) are coordinates of source points that located outside the domain.

Since seepage problem included moving boundary must be solved iteratively. Suppose that after the n^{th} iteration, value of $\varphi_i^{(n)}$ are known, values of $\varphi_i^{(n+1)}$ at $(n+1)^{\text{th}}$ iteration are to be determined. Therefore, the approximate solution of Eq. (1) can be represented by a linear combination of fundamental solution as

$$\varphi_i^{(n+1)} = \sum_{j=1}^{N} a_j^{(n+1)} G(P_i, Q_j) \qquad Q_j \in \hat{S} \tag{8}$$

where N be number of nodes in boundary domain. Substituting Eq. (8) into Eqs. (4) and (5) results in a system of equations:

$$\sum_{j=1}^{N} a_j^{(n+1)} G(P_i, Q_j) = f(x_i, y_i) \qquad (i = 1, 2, ..., N_1) \tag{9}$$

$$\sum_{j=1}^{N} a_j^{(n+1)} \left[n_x \frac{\partial}{\partial x} G(P_i, Q_j) + n_y \frac{\partial}{\partial y} G(P_i, Q_j) \right] = g(x_i, y_i)$$

$$(i = N_1 + 1, N_1 + 2, ..., N_1 + N_2) \tag{10}$$

where N_1 and N_2 are the number of nodes on boundary S_1 and S_2, respectively, and $N = N_1 + N_2$. Hence, $a_j^{(n+1)}$ can be determined.

Direction cosine n_x and n_y in Eq. (5) or Eq. (10) on free surface can be expressed as

$$n_x = \cos\alpha \tag{11}$$

$$n_y = \cos\beta \tag{12}$$

respectively, and shown in Figure 2.

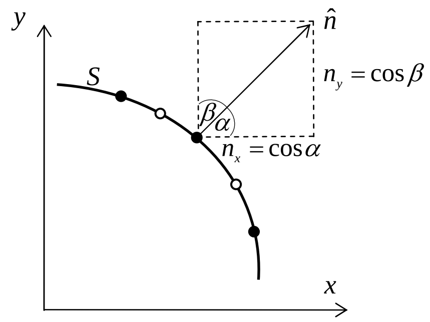

Figure 2. Direction cosine, and locations of boundary nodes (black circles) and central boundary nodes (white circles)

Therefore, each of iteration, direction cosines of moving boundary nodes are to be determined by using boundary nodes and central boundary nodes as displayed Figure 2. For central boundary nodes are interpolated by cubic spline interpolation (CBI) [8] and [9], when x-coordinate of those nodes specifically known. Moving boundary is also obtained by this interpolation technique. CBI is chosen because it uses third degree polynomials to connect the data points which often results in strikingly smooth curve fitting. For separation point is shown in Figure 1, it is calculated by second degree polynomials after free surface obtained for each of iteration.

Since the free surface has over specified boundary conditions, it will be determined iteratively by using initial guess for moving boundary as shown in Figure 3. Additionally, Figure3 shows positions of source points in the space coordinates. It can be seen that the number of source points is the number of boundary nodes (N). N source points have the space coordinates as

$$(\xi_i, \eta_i) = (x_i, y_i) + BF \cdot (n_{x,i}, n_{y,i}) \tag{13}$$

where BF is body factor constant, for this paper, let BF is equal to 1.0 to determine coordinate of source points. Each of source points is also located on an imaginary boundary, which is larger than the actual boundary.

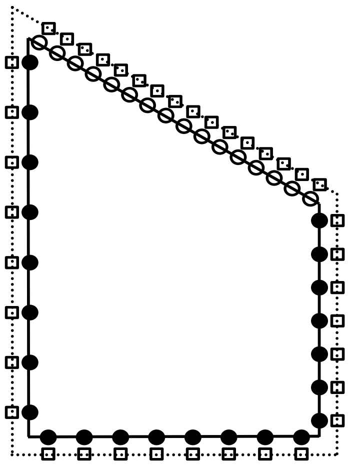

Figure 3. Sample of boundary nodes (black circles), initial moving boundary nodes (white circles), and source points (white squares)

The moving surface location is determined by checking the criterion of convergence as following

$$\varepsilon = \frac{\sqrt{\sum_{i=1}^{m} (\varphi_i^{n+1} - \varphi_i^{n})^2}}{\sqrt{\sum_{i=1}^{m} (\varphi_i^{n})^2}} \tag{13}$$

where the symbol m is the total number of boundary nodes on the moving surface, and the allowable tolerance used in this paper is 10^{-4} as same as Chen et al. [5]. The flowchart of iteration procedure is also displayed in Figure 4.

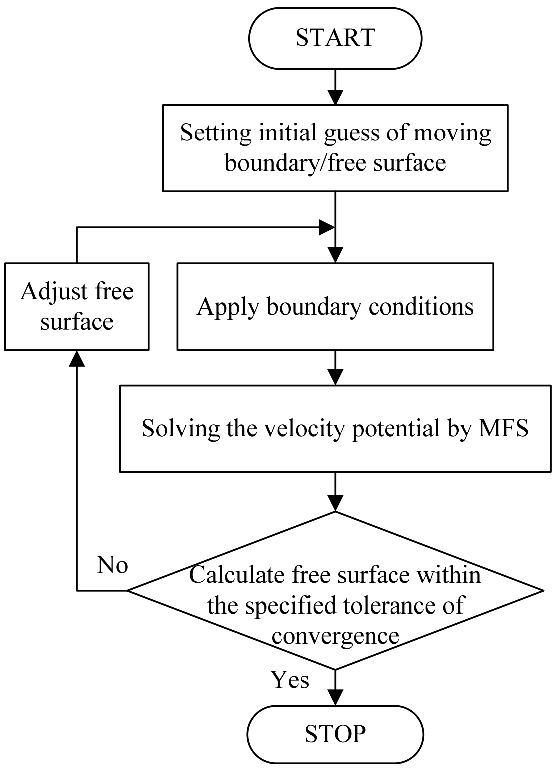

Figure 4. Flowchart of iteration procedure

4. Results and Discussion

Consider seepage problem, where the upper hydraulic head y_1 = 24, the lower hydraulic head y_2 = 4, and the width of dam x_1 = 16. There are 70 nodes uniformly distributed in the initial guess domain at assumed separation at y = 14 so that grid spacing is 1.0. The present numerical solutions of free surface are then compared with those of Aitchison [2], Westbrook [3], and Chen et al. [5] as shown Table 1 and Figure 5, respectively. The number of iterations of present method is 26.

Table 1. Moving boundary obtained by different methods

x (m)	MFS	Aitchison (FDM)	Westbrook (FEM)	Chen et al. (BEM)
1	23.75	23.74	23.64	23.74
2	23.41	23.41	23.32	23.40
3	23.03	23.02	23.06	23.01
4	22.59	22.59	22.52	22.52
5	22.12	22.12	22.12	22.09
6	21.60	21.60	21.55	21.57
7	21.04	21.04	21.07	21.00
8	20.44	20.43	20.36	20.39
9	19.79	19.78	19.81	19.73
10	19.08	19.08	19.07	19.02
11	18.32	18.31	18.26	18.24
12	17.50	17.48	17.45	17.39
13	16.59	16.57	16.45	16.45
14	15.58	15.54	15.51	15.39
15	14.40	14.39	14.33	14.09
16	12.88	12.79	N/A	12.68

For the free surface problem, it is one kind of inverse problem since boundary is not known in a priori. As a result, the analytical solution of this problem is difficult to find. But Aitchison's solution is based on the semi-analytical by using the complex variable method, so these data are more accurate and believable than other numerical results. Therefore, it is chosen to compare with MFS solutions. Results indicate that MFS is capable to calculate free surface agree with other methods.

The separation point at $x = 16$ is interesting and important since a singular point due to the intersection of the free surface and seepage surface. In addition, this point presents an important role in term of dam stability. It is predicted by MFS and compared with other methods as shown Table 2.

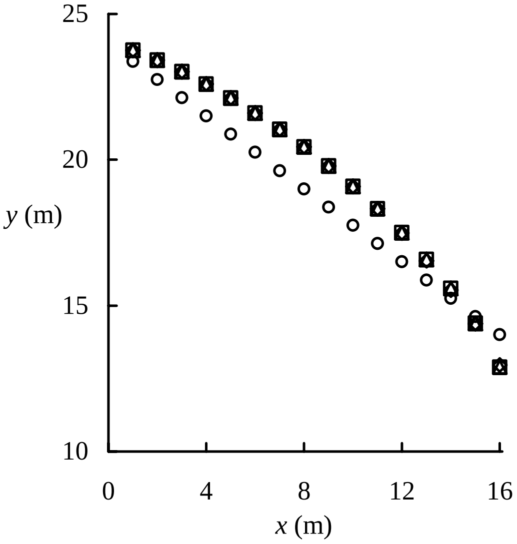

Figure 5. Locations of initial guess free surface (circles), free surface at iteration = 10 (diamonds), free surface at iteration = 20 (triangles), and final free surface at iteration = 26 (squares).

Table 2. The separation point calculated by different methods

Reference	Height (m)
MFS	12.88
Aitchison (FDM)	12.79
Westbrook (FEM)	N/A
Chen et al. (BEM)	12.68

5. Conclusion

In this paper, it is shown how to use MFS to solve the problem of two-dimensional steady-state isotropic seepage flow. The free surface or moving boundary, and separation point can be obtained. Although it is only considered solving dam problem, MFS can be applied to more general moving boundary problems.

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