Comparative Performance Between Discrete Kirchhoff Triangular and Standard Rectangular Plate Bending Elements

Parinya Boonmarlert[1], Suthee Traivivatana[2], Pichayen Bhothikhun[3] and Pramote Dechaumphai[4]*

[1,2,3,4] Department of Mechanical Engineering, Chulalongkorn University, 254 Phyathai Road, Patumwan, Bangkok 10330, Thailand,
Tel: 0-2218-6621, Fax: 0-2218-6621 *E-mail: fmepdc@eng.chula.ac.th

Abstract

Comparative performance between the discrete Kirchhoff triangular element and the standard rectangular element for plate bending analysis is presented. The discrete Kirchhoff triangular element has three nodes, while the standard rectangular element contains four nodes. The Galerkin finite element method is employed to derive the corresponding finite element equations and their matrices for both element types. Several examples that have exact solutions are used to evaluate their performances. Results show that the three-node discrete Kirchhoff triangular element performs very well as compared to the standard four-node rectangular element.

Keywords: Finite element method, Discrete Kirchhoff triangle (DKT), Transverse deflection

1. Introduction

It has been known that the three-node triangular element can provide high flexibility in the construction of finite models for complex geometry in two-dimensions. A finite element mesh can easily be constructed with different element sizes. The triangular elements can also be combined with an adaptive meshing technique to provide high solution accuracy at reduced computational effort. An adaptive finite element mesh normally contains small clustered elements in the regions of high stress gradient to provide accurate solution. Larger elements are constructed in the other regions to reduce the total number of unknowns and thus the computational time. However, the use of the standard three-node triangular element does not give high solution accuracy as compared to the other standard four-node element, such as the rectangular element.

In this paper, another three-node triangular element, so called the Discrete Kirchhoff Triangle (DKT) element [1], is investigated. The element has three unknowns per node as the standard triangular element. The element, however, can provide higher solution accuracy, because several key assumptions have been made in the development of the element. The performance of the DKT element will be compared with those of the standard four-node rectangular element [2]. The paper starts from explaining the governing equation for the transverse deflection of plate. The corresponding finite element equations and the associated element matrices for both the DKT element and the standard rectangular element are presented. Finally, the performance of the DKT element and the standard rectangular element are evaluated by solving several examples. The predicted solutions are compared with the exact solutions of the problems.

2. Governing Equations

The equation for the transverse deflection, w, in the z-direction normal to the x-y plane of a thin plate with a constant thickness of t whose middle plane is coincident with the x-y plane, is given by the equilibrium equation in the form [3],

$$D\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right) = p(x,y) \qquad (1)$$

where $p(x,y)$ is the applied lateral load normal to the plate and D is the bending rigidity. The bending rigidity is defined by,

$$D = \frac{Et^3}{12(1-\nu^2)} \qquad (2)$$

where E is the modulus of elasticity, ν is the Poisson's ratio.

Depending on the problems, the boundary conditions for plates involve the transverse deflection, w, and its derivatives. Typical boundary conditions are simply-supported, clamped and free edges, including the specified deflection. These different boundary conditions, depending on the given problems, are imposed prior to solving for solutions.

3. Derivation of Finite Element Equations

Derivation of the finite element equations for the Discrete Kirchhoff Triangle (DKT) element and the standard rectangular element are briefly described herein.

3.1 Discrete Kirchoff Triangle (DKT)

The derivation of the three-node DKT element equations is based on the following assumptions: 1) both the x- and y-twist angles vary quadratically over the element, 2) the transverse shears are zero at the tip nodes, 3) the transverse deflection is in form of a cubic function over the element, and 4) the twist angles normal to the element sides vary linearly. The finite element equations are derived by applying the method of weighted residuals to the plate bending equation Eq. (1) leading to the finite element equations in the form,

$$[K]\{\delta\} \quad = \quad \{F\} \tag{3}$$

where the vector, $\{\delta\}$, contains the element nodal unknowns of transverse deflection and the rotations. Each node has a transverse deflection in the z-direction and the two rotations about the x- and y-directions. Thus there are nine degrees of freedom per element. The nodal force vector, $\{F\}$, may be due to the applied loads such as the concentrated load and the pressure load. The stiffness matrix and the load vector due to the applied pressure are defined by,

$$[K] \quad = \quad \int_A [B]^T [D][B]\, dA \tag{4a}$$

$$\{F\} \quad = \quad \int_A [N]^T p\, dA \tag{4b}$$

where the strain-displacement interpolation matrix, $[B]$, is defined by,

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{31}\left[\dfrac{\partial H_x}{\partial \xi}\right] + y_{12}\left[\dfrac{\partial H_x}{\partial \eta}\right] \\ -x_{31}\left[\dfrac{\partial H_y}{\partial \xi}\right] - x_{12}\left[\dfrac{\partial H_y}{\partial \eta}\right] \\ -x_{31}\left[\dfrac{\partial H_x}{\partial \xi}\right] - x_{12}\left[\dfrac{\partial H_x}{\partial \eta}\right] + y_{31}\left[\dfrac{\partial H_y}{\partial \xi}\right] + y_{12}\left[\dfrac{\partial H_y}{\partial \xi}\right] \end{bmatrix} \tag{5}$$

where

$$\left\{\frac{\partial H_x}{\partial \xi}\right\} = \begin{Bmatrix} p_6(1-2\xi) + \eta(p_5 - p_6) \\ q_6(1-2\xi) - \eta(q_5 + q_6) \\ -4 + 6(\xi+\eta) + r_6(1-2\xi) - \eta(r_5 + r_6) \\ -p_6(1-2\xi) + \eta(p_4 + p_6) \\ q_6(1-2\xi) + \eta(q_4 - q_6) \\ -2 + 6\xi + r_6(1-2\xi) + \eta(r_4 - r_6) \\ -\eta(p_4 + p_5) \\ \eta(q_4 - q_5) \\ \eta(r_4 - r_5) \end{Bmatrix} \tag{6a}$$

$$\left\{\frac{\partial H_y}{\partial \xi}\right\} = \begin{Bmatrix} t_6(1-2\xi) + \eta(t_5 - t_6) \\ 1 + r_6(1-2\xi) - \eta(r_5 + r_6) \\ -q_6(1-2\xi) + \eta(q_5 + q_6) \\ -t_6(1-2\xi) + \eta(t_4 + t_6) \\ -1 + r_6(1-2\xi) - \eta(r_4 - r_6) \\ -q_6(1-2\xi) - \eta(q_4 - q_6) \\ -\eta(t_4 + t_5) \\ \eta(r_4 - r_5) \\ -\eta(q_4 - q_5) \end{Bmatrix} \tag{6b}$$

$$\left\{\frac{\partial H_x}{\partial \eta}\right\} = \begin{Bmatrix} -p_5(1-2\eta) + \xi(t_5 - t_6) \\ q_5(1-2\eta) - \xi(q_5 + q_6) \\ -4 + 6(\xi+\eta) + r_5(1-2\eta) - \xi(r_5 + r_6) \\ \xi(p_4 + p_6) \\ \xi(q_4 - q_6) \\ \xi(r_4 - r_6) \\ p_5(1-2\eta) - \xi(p_4 + p_5) \\ q_5(1-2\eta) + \xi(q_4 - q_5) \\ -2 + 6\eta + r_5(1-2\eta) + \xi(r_4 - r_5) \end{Bmatrix} \tag{6c}$$

$$\left\{\frac{\partial H_y}{\partial \eta}\right\} = \begin{Bmatrix} -t_5(1-2\eta) + \xi(t_5 - t_6) \\ 1 + r_5(1-2\eta) - \xi(r_5 + r_6) \\ -q_5(1-2\eta) + \xi(q_5 + q_6) \\ \xi(t_4 + t_6) \\ \xi(r_4 - r_6) \\ -\xi(q_4 - q_6) \\ t_5(1-2\eta) - \xi(t_4 + t_5) \\ -1 + r_5(1-2\eta) + \xi(r_4 - r_5) \\ -q_5(1-2\eta) - \xi(q_4 - q_5) \end{Bmatrix} \tag{6d}$$

The coefficients p_k, q_k, r_k and t_k, k = 4, 5, 6 depend on the element shape and are given by,

$$p_k = \frac{-6x_{ij}}{\ell_{ij}^2} \tag{7a}$$

$$q_k = \frac{3x_{ij}y_{ij}}{\ell_{ij}^2} \tag{7b}$$

$$r_k = \frac{3y_{ij}^2}{\ell_{ij}^2} \tag{7c}$$

$$t_k = \frac{-6y_{ij}}{\ell_{ij}^2} \tag{7d}$$

$$\ell = \sqrt{x_{ij}^2 + y_{ij}^2} \tag{7e}$$

where the coefficients x_{ij} and y_{ij}, i, j = 1, 2, 3 are defined in terms of element nodal coordinates by,

$$x_{ij} = x_i - x_j \qquad y_{ij} = y_i - y_j \tag{8}$$

The matrix $[D]$ in Eq.(4) is the plate material stiffness matrix defined by,

$$[D] = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{9}$$

The above finite element matrices are in closed-form so that they can be implemented in the computer program directly.

3.2 Standard rectangular element

The four-node rectangular element is widely used in the plate bending analysis because it can provide good solution accuracy and its formulation is simple. The element has dimensions of a and b, in the x- and y-directions, respectively, with the thickness of h. Each node has the transverse deflection, w, and the two rotations, θ_x and θ_y. Thus each element contains twelve degrees of freedom. The procedure for deriving the finite element equation is similar to that explained in Section 3.1. The derivation of the element equations starts from using the standard rectangular element interpolation functions, $N_i(x,y)$, given by,

$$N_1 = 1 - \frac{3x^2}{a^2} - \frac{xy}{ab} - \frac{3y^2}{b^2} + \frac{2x^3}{a^3} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2} + \frac{2y^3}{b^3} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab^3} \tag{10a}$$

$$N_2 = y - \frac{xy}{a} - \frac{2y^2}{b} + \frac{2xy^2}{ab} + \frac{y^3}{b^2} - \frac{xy^3}{ab^2} \tag{10b}$$

$$N_3 = -x + \frac{2x^2}{a} + \frac{xy}{b} - \frac{x^3}{a^2} - \frac{2x^2y}{ab} + \frac{x^3y}{a^2b} \tag{10c}$$

$$N_4 = \frac{3x^2}{a^2} + \frac{xy}{ab} - \frac{2x^3}{a^3} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3} \tag{10d}$$

$$N_5 = \frac{xy}{a} - \frac{2xy^2}{ab} + \frac{xy^3}{ab^2} \tag{10e}$$

$$N_6 = \frac{x^2}{a} - \frac{x^3}{a^2} - \frac{x^2y}{ab} + \frac{x^3y}{a^2b} \tag{10f}$$

$$N_7 = -\frac{xy}{ab} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab^3} \tag{10g}$$

$$N_8 = -\frac{xy^2}{ab} + \frac{xy^3}{ab^2} \tag{10h}$$

$$N_9 = \frac{x^2y}{ab} - \frac{x^3y}{a^2b} \tag{10i}$$

$$N_{10} = \frac{xy}{ab} + \frac{3y^2}{b^2} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} - \frac{2y^3}{b^3} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3} \tag{10j}$$

$$N_{11} = -\frac{y^2}{b} + \frac{xy^2}{ab} + \frac{y^3}{b^2} - \frac{xy^3}{ab^2} \tag{10k}$$

$$N_{12} = -\frac{xy}{b} + \frac{2x^2y}{ab} - \frac{x^3y}{a^2b} \tag{10l}$$

The full details of the method formulation and the element matrices for both the DKT element and the standard rectangular element are presented in Ref. [2].

4. Results

To evaluate the performance of the DKT element as compared to the standard rectangular element, the several examples that have exact solutions are used as presented below.

4.1 Simply supported square plate under uniform distributed load

A square 2×2 m simply supported plate with a thickness of 0.01 m, subjected to a uniform distributed load of 1,200 N/m^2, is shown in Fig. 1. The plate is assumed to have the modulus of elasticity of 7.2×10^{10} N/m^2 and the Poisson's ratio of 0.25. The exact transverse deflection can be derived and is given by [3,4],

$$w(x,y) = \frac{16pa^4}{\pi^6 D}\sum_{m=1,3..}^{\infty}\sum_{n=1,3..}^{\infty}\frac{\cos\left(\frac{m\pi x}{2a}\right)\cos\left(\frac{n\pi y}{2a}\right)}{mn(m^2+n^2)^2} \tag{11}$$

The results of the transverse deflection obtained from the DKT element and the rectangular element are shown in Figs. 2-3, respectively. Both of the finite element models have the same number of unknowns. Figure 4 shows the predicted transverse deflections along the x-direction obtained from both the element types as compared to the exact solution. The predicted maximum transverse deflections including their percentage errors from the exact solution are also shown in Table 1. The table shows the DKT element performs very well and provides good solution accuracy as compared to that of the standard four-node rectangular element.

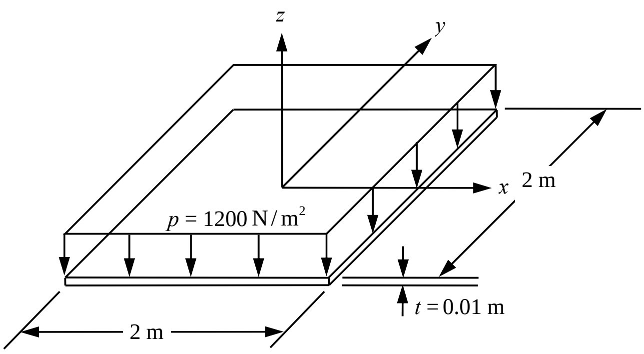

Figure 1. Problem statement of a simply supported square plate subjected to a uniformly distributed load.

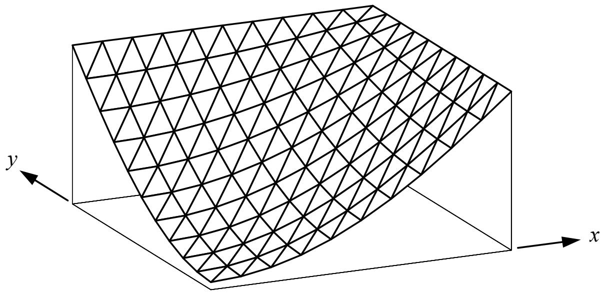

Figure 2. Predicted plate transverse deflection using the three-node DKT elements.

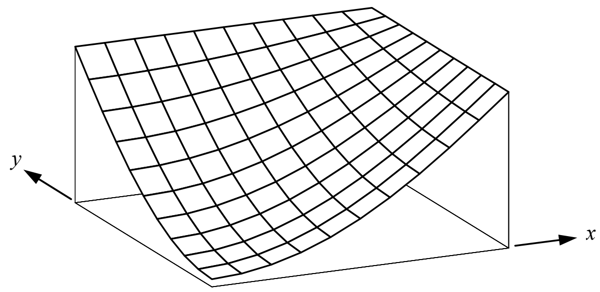

Figure 3. Predicted plate transverse deflection using the four-node rectangular elements.

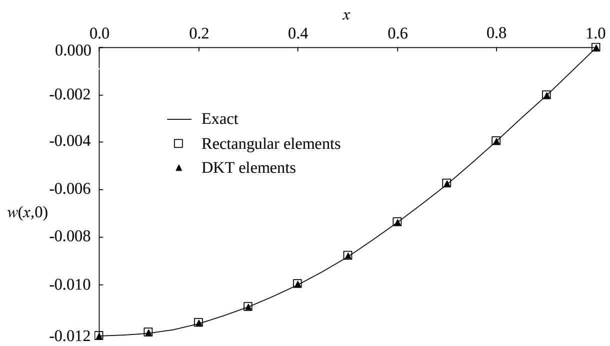

Figure 4. Comparative transverse deflections from the two finite element models with the exact solution along x-direction for simply supported square plate under uniform distributed load.

Table 1. Comparison of the predicted maximum transverse deflections and percentage errors for simply supported square plate under uniform distributed load.

	w_{max} (m)	|error (%)|
Exact	-0.012187	-
DKT element	-0.012170	0.13
Rectangular element	-0.012218	0.25

4.2 Simply supported square plate under concentrated load

The problem statement of the second problem is similar to the previous one except that the uniform distributed load is replaced by the concentrated load F = 1,200 N at the center of the plate. The exact solution for transverse deflection can be derived in form of the infinite series as [3,4],

$$w(x,y) = \frac{4Fa^2}{\pi^4 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)}{(m^2+n^2)^2} \tag{12}$$

where a is a plate length.

The comparison of the transverse deflection along the x-direction obtained from the DKT and the standard rectangular element with the exact solution is shown in Fig. 5. The predicted maximum transverse deflections including their percentage errors from the exact solution are also shown in Table 2. The table shows the DKT element can provide good solution accuracy as compared to the standard four-node rectangular element.

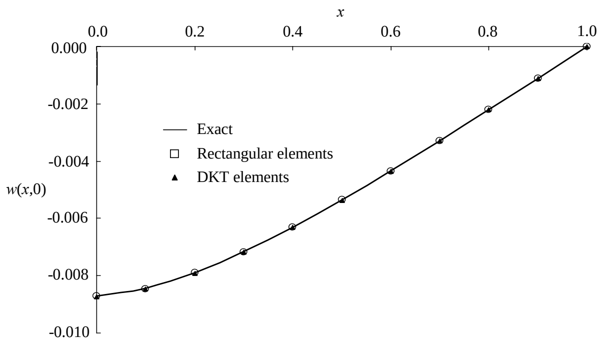

Figure 5. Comparative transverse deflections from the two finite element models with the exact solution along x-direction for simply supported square plate under concentrated load.

Table 2. Comparison of the predicted maximum transverse deflections and percentage errors for simply supported square plate under concentrated load.

	w_{max} (m)	|error (%)|
Exact	-0.008701	-
DKT element	-0.008708	0.08
Rectangular element	-0.008734	0.38

4.3 Rectangular plate with two opposite edges simply supported and one clamp supported under uniform distributed load

A rectangular plate with the dimensions of 1×1.5 m and a thickness of 0.01 m, subjected to a uniform distributed load of 10,000 N/m^2, is shown in Fig. 6. The plate is assumed to have the modulus of elasticity of 1.092×10^{11} N/m^2 and the Poisson's ratio of 0.3. The edges along x = 0 and 1 have the simply supported boundary condition, while the edge along y = 0 is clamped and the edge along y = 1.5 is free. The predicted plate deflections obtained from the DKT element and the standard rectangular element are shown in Figs. 7-8. The predicted maximum transverse deflections including their percentage errors from the exact solution [3,4] are shown in Table 3.

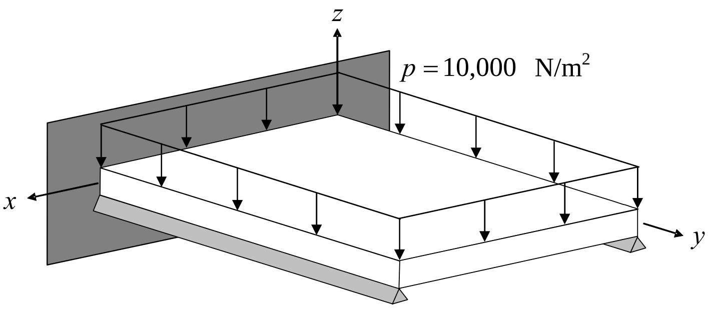

Figure 6. Problem statement of a rectangular plate with two opposite edges simply supported and one clamp supported under uniform distributed load.

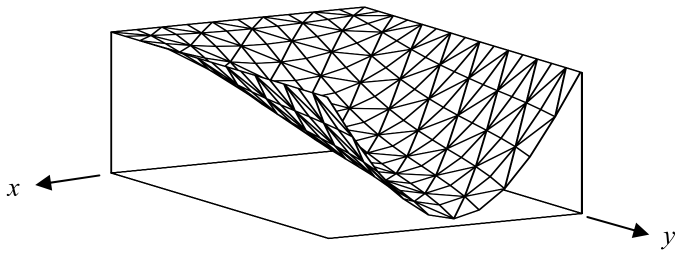

Figure 7. Predicted plate deformation by using three-node DKT elements.

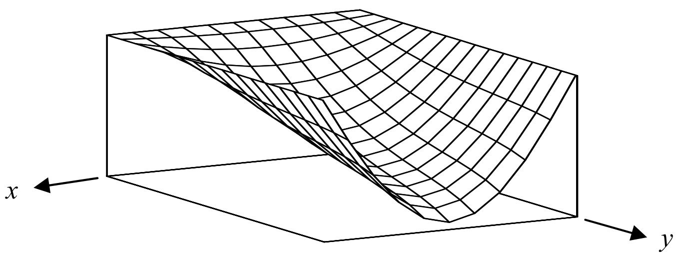

Figure 8. Predicted plate deformation by using four-node rectangular elements.

Table 3. Comparison of the predicted maximum transverse deflections and percentage errors for a rectangular plate with two opposite edges simply supported and one clamp supported under uniform distributed load.

	w_{max} (m)	|error (%)|
Exact	-0.014100	-
DKT element	-0.014041	0.42
Rectangular element	-0.014035	0.46

5. Conclusion

Comparative performance between the discrete Kirchhoff triangular (DKT) element and the standard rectangular element for plate bending analysis was presented. The finite element equations for both the DKT element and the standard rectangular element were derived by using the method of weighted residuals. All finite element matrices were derived in closed-form and the corresponding computer programs were developed. Three plate bending examples with exact solutions were used to evaluate the performance of the DKT element. The results show that the DKT element performs very well as compared to the standard four-node rectangular element.

Acknowledgments

The authors are pleased to acknowledge the Thailand Research Fund (TRF) for supporting this research work.

References

[1] Batoz, J. L., Bathe, K. J., and Ho, L. W., 1980. A Study of Three-Node Triangular Plate Bending Elements. International Journal for Numerical Methods in Engineering, Vol. 15, pp. 1771-1812.

[2] Dechaumphai, P., 2007. Finite Element Method in Engineering. 4th Ed., Chulalongkorn University Press, Bangkok.

[3] Timoshenko, S. and Krieger, S. W., 1970. Theory of Plates and Shells. 3rd Ed., McGraw-Hill, Singapore.

[4] Ugural, A. C., 1999. Stresses in Plates and Shells. 2nd Ed., McGraw-Hill, Singapore.