# Use of Space-Time Nine-Node Rectangular Elements for Solving Transient One-Dimensional Conduction Heat Transfer Problems

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**Abstract**

A finite element method for transient one-dimensional conduction heat transfer problems is presented. Space-time nine-node rectangular elements were employed with the weighted residuals method in the solution procedures. The corresponding finite element computer program was developed and verified by example problems with and without instantaneous pulse sources. The pseudo-time intervals were inserted in the time marching process when instantaneous pulse sources exist. All numerical results showed excellent agreements with analytical solutions.

**Keywords**: finite element, transient, conduction, nine-node rectangular elements.

## 1. Introduction

Steady and transient conduction heat transfer problems are basic problems in the subject of heat transfer. They were studied extensively from the origin of the science of heat transfer. Ozisik [1] compiled the standard analytical methods for exact solutions of these problems. Sturm-Liouville theory, Duhamel's theorem, Green's function, Laplace's transform, and integral-transform technique were mentioned. Also, the finite difference method was introduced. Zienkiewicz and Taylor [2] summarized the approximation methods for time-dependent field problems. The method of semi-discretization, single-step methods, and multi-step methods were explained. All of the methods in these two references can fulfill the general purposes of engineers and researchers in solving conduction heat transfer problems.

Nowadays, the approximation methods are very popular because they can cope with the difficulties that make the analytical methods fail or be clumsy such as complex geometry, various boundary conditions, the occurrence of pulse sources, and nonhomogeneities. In the last few decades, countless researches on the approximations of conduction heat transfer equation by finite element method have been published. The concept of weighted-residuals finite element has been utilized only in spatial domain while the recurrence formulae have been employed in time domain to reduce the economical efforts in obtaining the numerical solutions. The recent examples of attacking the heat conduction problems by the finite difference in time domain are the papers published by Ilinca and Hetu [3], Chen and Tong [4], and Wang and Mai [5]. However, the concept of weighted-residuals method can be directly applied to spatial and time domain.

The main objective of this paper is to present a finite element method for solving transient one-dimensional conduction heat transfer problems based on space-time nine-node rectangular elements as the representative application of weighted-residuals finite element to spatial and time domain. The second-order discretization on spatial and time domain can be achieved simultaneously. The corresponding finite element computer program that can be executed on standard personal computer was developed. Numerical solutions obtained from the developed finite element computer program were verified by comparing with the analytical solutions provided by VanSant [6].

## 2. Governing Equations

The differential equation of heat conduction for a stationary, homogeneous, isotropic solid occupying the region $\Omega$ with heat generation within its for $t > 0$ [1] is

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} + q''' = \rho c_p \frac{\partial T}{\partial t} \tag{1}$$

where $q_x$, $q_y$, $q_z$ are heat flux in $x$-, $y$-, and $z$-direction, $q'''$ is rate of heat generation per unit volume, $\rho$ is the solid density, $c_p$ is the specific heat at constant pressure of solid, $T$ is the solid temperature, and $\Omega$ is the spatial domain occupied by the conducting solid.

From Fourier's law of conduction for isotropic materials, heat fluxes in $x$-, $y$-, and $z$- direction are

$$q_x = -k\frac{\partial T}{\partial x},\ q_y = -k\frac{\partial T}{\partial y},\ q_z = -k\frac{\partial T}{\partial z} \tag{2}$$

where $k$ is the thermal conductivity of solid.

A set of boundary conditions and an initial condition must be prescribed before finding the solutions. The initial condition specifies the temperature distribution within the spatial domain at the origin of time coordinate.

$$T(x,y,z,t) = F(x,y,z) \tag{3}$$

on region $\Omega$ at $t=0$.

The boundary conditions specify one or combination of the following conditions on the boundary surface of spatial domain.

Boundary condition of first kind:

$$T(x,y,z,t)=f(x,y,z,t) \tag{4}$$

on $\Gamma$ for $t>0$, where $\Gamma$ is the boundary surface of spatial domain.

Boundary condition of second kind:

$$k\frac{\partial T}{\partial n}=f(x,y,z,t) \tag{5}$$

on $\Gamma$ for $t>0$, where $n$ is the outward normal coordinate.

Boundary condition of third kind:

$$k\frac{\partial T}{\partial n}+hT=hT_\infty \tag{6}$$

on $\Gamma$ for $t>0$, where $h$ is the convection coefficient and $T_\infty$ is the ambient fluid temperature.

For one-dimensional energy transfer, the differential equation of heat conduction becomes

$$-\frac{\partial q_x}{\partial x}+q'''=\rho c_p\frac{\partial T}{\partial t} \tag{7}$$

in $0<x<l$ for $t>0$.

The set of boundary conditions and an initial condition described for this case are:-

Boundary conditions:

$$k_i\frac{\partial T}{\partial n_i}+h_iT=\delta_{ij}f_j(x,y,z,t) \tag{8}$$

on $\Gamma_i$ where $\delta_{ij}$ is Kronecker delta, $i,j=1,2$.

Initial condition:

$$T(x,t)=F(x) \tag{9}$$

for $0\le x\le l$ at $t=0$.

**3. Finite Element Formulation**

To distinguish the concept of directly applied weighted-residuals methods to the conduction heat transfer equation from other well-established standard methods, the nine-node rectangular elements are selected for this problem. Figure 1 shows that a nine-node rectangular element is originated from a series of one three-node line element in time domain. Node numbering is detailed in Figure 2. The interpolation functions for element of this type are constructed as follow [7].

The shape functions of line element with length $a$ are

$$L_{1x}=1-x/a,\quad L_{2x}=x/a \tag{10}$$

The parabolic interpolation functions for steady problems are constructed by using the combination of the complete set of second-order polynomials in the form

$$T(x)=\alpha_1L_{1x}^2+\alpha_2L_{1x}L_{2x}+\alpha_3L_{2x}^2 \tag{11}$$

If the problems are unsteady, the complete set of second-order polynomials in spatial and time domain is

$$\begin{aligned}T(x,t)&=\beta_1L_{1x}^2L_{1t}^2+\beta_2L_{1x}^2L_{1t}L_{2t}+\beta_3L_{1x}^2L_{2t}^2\\&+\beta_4L_{1x}L_{2x}L_{1t}^2+\beta_5L_{1x}L_{2x}L_{1t}L_{2t}+\beta_6L_{1x}L_{2x}L_{2t}^2\\&+\beta_7L_{2x}^2L_{1t}^2+\beta_8L_{2x}^2L_{1t}L_{2t}+\beta_9L_{2x}^2L_{2t}^2\end{aligned} \tag{12}$$

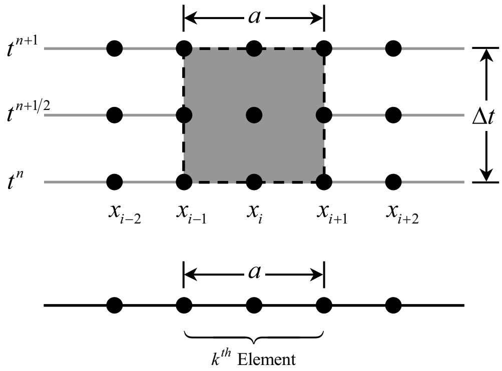


Figure 1. A nine-node rectangular element.

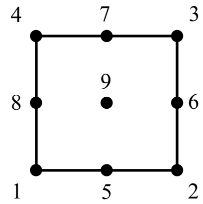


Figure 2. Node numbering of a nine-node rectangular element.

where

$$L_{1t}=1-t/\Delta t,\quad L_{2t}=t/\Delta t \tag{13}$$

After finding nine coefficients from nine conditions at each node and rearranging, the temperature distribution on an element is

$$T(x,t)=N_iT_i \tag{14}$$

where $i=1,2,3,...,9$ and

$$\begin{aligned}N_1(x,t)&=L_{1x}(L_{1x}-L_{2x})\cdot L_{1t}(L_{1t}-L_{2t})\\N_2(x,t)&=L_{2x}(L_{2x}-L_{1x})\cdot L_{1t}(L_{1t}-L_{2t})\\N_3(x,t)&=L_{2x}(L_{2x}-L_{1x})\cdot L_{2t}(L_{2t}-L_{1t})\\N_4(x,t)&=L_{1x}(L_{1x}-L_{2x})\cdot L_{2t}(L_{2t}-L_{1t})\\N_5(x,t)&=4L_{1x}L_{2x}L_{1t}(L_{1t}-L_{2t})\\N_6(x,t)&=4L_{2x}L_{1t}L_{2t}(L_{2x}-L_{1x})\\N_7(x,t)&=4L_{1x}L_{2x}L_{2t}(L_{2t}-L_{1t})\\N_8(x,t)&=4L_{1x}L_{1t}L_{2t}(L_{1x}-L_{2x})\\N_9(x,t)&=16L_{1x}L_{2x}L_{1t}L_{2t}\end{aligned} \tag{15}$$

The method of weighted-residuals is applied by weighting the differential equation of one-dimensional heat conduction with the interpolation functions of nine-node rectangular element to obtain the equations

$$-\int_0^{\Delta t}\left(\iiint_{\Omega_e}N_i\frac{\partial q_x}{\partial x}d\Omega_e\right)dt+\int_0^{\Delta t}\left(\iiint_{\Omega_e}N_iq'''d\Omega_e\right)dt=\int_0^{\Delta t}\left(\iiint_{\Omega_e}N_i\rho c_p\frac{\partial T}{\partial t}d\Omega_e\right)dt \tag{16}$$

where $i=1,2,3,...,9$.

After applying integration by parts and Gauss theorem, equation (16) becomes

$$\int_0^{\Delta t}(N_i q_x A)_{left}\,dt-\int_0^{\Delta t}(N_i q_x A)_{right}\,dt$$
$$+\int_0^{\Delta t}\left(\int_0^{a}\frac{\partial N_i}{\partial x}q_x A dx\right)dt+\int_0^{\Delta t}\left(\int_0^{a}N_i q''' A dx\right)dt$$
$$=\int_0^{\Delta t}\left(\int_0^{a}N_i\rho c_p\frac{\partial T}{\partial t}A dx\right)dt \tag{17}$$

where $A$ is the cross-sectional area at any point of the element, $(q_x A)_{left}$ is the amount of heat transfer at the left end of element, and $(q_x A)_{right}$ is the amount of heat transfer at the right end of element.

The amount of heat transfer at both ends of element can be computed with the aid of the constructed interpolation functions in the following manners.

$$(q_x A)_{left}=(q_x A)_1 N_{1t}(0,t)$$
$$+(q_x A)_8 N_{8t}(0,t)+(q_x A)_4 N_{4t}(0,t) \tag{18}$$
$$(q_x A)_{right}=(q_x A)_2 N_{2t}(a,t)$$
$$+(q_x A)_6 N_{6t}(a,t)+(q_x A)_3 N_{3t}(a,t) \tag{19}$$

Substituting Fourier's law of conduction into equation (17) and rearranging result in the finite element equations in the form

$$[K_c]_e\{T\}_e+[K_k]_e\{T\}_e=\{Q_g\}_e+\{Q_s\}_e \tag{20}$$

where $[K_c]_e$ is the element capacitance matrix, $[K_k]_e$ is the element conductance matrix, $\{T\}_e$ is the vectors of element unknown temperatures, $\{Q_g\}_e$ is the element load vectors from heat generation, and $\{Q_s\}_e$ is the element load vectors from heat transfer at the boundary surface.

Assuming that all solid properties are constants and all elements in computational domain have constant cross-sectional areas, the closed-forms of the element matrices are

$$[K_c]_e=\frac{\rho c_p A a}{180}\begin{bmatrix}-12&3&1&-4&-6&-4&-2&16&8\\3&-12&-4&1&-6&16&-2&-4&8\\-1&4&12&-3&2&-16&6&4&-8\\4&-1&-3&12&2&4&6&-16&-8\\-6&-6&-2&-2&-48&8&-16&8&64\\4&-16&16&-4&-8&0&8&0&0\\2&2&6&6&16&-8&48&-8&-64\\-16&4&-4&16&-8&0&8&0&0\\-8&-8&8&8&-64&0&64&0&0\end{bmatrix} \tag{21}$$

$$[K_k]_e=\frac{kA}{90}\cdot\frac{\Delta t}{a}\begin{bmatrix}28&4&-1&-7&-32&2&8&14&-16\\&28&-7&-1&-32&14&8&2&-16\\&&28&4&8&14&-32&2&-16\\&&&28&8&2&-32&14&-16\\&&&&64&-16&-16&-16&32\\&&&&&112&-16&16&-128\\&&&&&&64&-16&32\\&&&&&&&112&-128\\\text{Sym}&&&&&&&&256\end{bmatrix} \tag{22}$$

$$\{Q_g\}_e=\frac{q''' A a}{36}\Delta t\{1\quad 1\quad 1\quad 1\quad 4\quad 4\quad 4\quad 4\quad 16\}^T \tag{23}$$

$$\{Q_s\}_e=\frac{\Delta t}{30}\begin{Bmatrix}4(q_x A)_1+2(q_x A)_8-(q_x A)_4\\4(q_x A)_2+2(q_x A)_6-(q_x A)_3\\-(q_x A)_2+2(q_x A)_6+4(q_x A)_3\\-(q_x A)_1+2(q_x A)_8+4(q_x A)_4\\0\\2(q_x A)_2+16(q_x A)_6+2(q_x A)_3\\0\\2(q_x A)_1+16(q_x A)_8+2(q_x A)_4\\0\end{Bmatrix} \tag{24}$$

When both ends of the element are subjected to convection by the ambient fluid, two terms are incorporated into the finite element equations to yield

$$[K_c]_e\{T\}_e+[K_k]_e\{T\}_e+[K_h]_e\{T\}_e$$
$$=\{Q_g\}_e+\{Q_s\}_e+\{Q_h\}_e \tag{25}$$

where $[K_h]_e$ is the element convection matrix and $\{Q_h\}_e$ is the element load vectors from convection. The closed-forms of these two matrices are

For left end,

$$[K_h]_e=\frac{hA}{30}\Delta t\begin{bmatrix}4&0&0&-1&0&0&0&2&0\\&0&0&0&0&0&0&0&0\\&&0&0&0&0&0&0&0\\&&&4&0&0&0&2&0\\&&&&0&0&0&0&0\\&&&&&0&0&0&0\\&&&&&&0&0&0\\&&&&&&&16&0\\\text{Sym}&&&&&&&&0\end{bmatrix} \tag{26}$$

$$\{Q_h\}_e=\frac{hAT_\infty}{6}\Delta t\{1\quad 0\quad 0\quad 1\quad 0\quad 0\quad 0\quad 4\quad 0\}^T \tag{27}$$

For right end,

$$[K_h]_e = \frac{hA}{30}\Delta t \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 4 & -1 & 0 & 0 & 2 & 0 & 0 & 0 \\ & & 4 & 0 & 0 & 2 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 \\ & & & & & 16 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 \\ & & & & & & & 0 & 0 \\ \text{Sym} & & & & & & & & 0 \end{bmatrix} \tag{28}$$

$$\{Q_h\}_e = \frac{hAT_\infty}{6}\Delta t \{0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0\}^T \tag{29}$$

In fact, the convection heat transfer can occur only at the ends of the computational domain. Thus, the finite element matrices associated with convection should be computed and incorporated after the assembling of the system equations. The final form of the system equations can be written as

$$[K]_{sys}\{T\}_{sys} = \{Q_g\}_{sys} + \{Q_s\}_{sys} + \{Q_h\}_{sys} \tag{30}$$

where $[K]_{sys} = [K_c]_{sys} + [K_k]_{sys} + [K_h]_{sys}$, $\{Q_g\}_{sys}$ is the system load vector from heat generation, $\{Q_s\}_{sys}$ is the system load vector from point heat sources, and $\{Q_h\}_{sys}$ is the system load vector from convection.

After imposing the boundary conditions and the initial condition, the preconditioning conjugate gradient method [8] is applied to solve the system equations with the stopping criteria of $10^{-6}$ in 2-norm relative residuals.

The described procedures are used in the developing of finite element computer program that can be executed on standard personal computers. The numerical solutions obtained from the developed finite element computer program are verified by comparing with the analytical solutions given by VanSant [6] in the following section.

## 4. Numerical Results and Discussions

The solid properties used in all calculations are the properties of AISI 4030 steel at 300 *K* [6] except the specific heat at constant pressure, $c_p$, which is one-thousandth of real value. The cross-sectional area and the length of the conducting solid are 0.1 $m^2$ and 1 *m*. This set of numerical values results in the problems that reach to steady state rapidly. Six example problems with 100 elements per time step are carefully selected to validate the developed finite element computer program. In first three example problems, no pulse sources exist and the rate of heat generation is set to be 10 $W/m^3$. In last three example problems, there is no heat generation but the instantaneous pulse sources occur at the beginning of the time coordinate. The pulse strength $Q_0''$ is set to be 40 $J/m^2$ and $Q_0'''$ is 200 $J/m^2$. All example problems are detailed below.

<u>Case I</u>

The physical situation of transient heat conduction with specified temperatures at both ends is shown in Figure 3.

Initial Condition:

$$T(x,0) = T_0 = 0 \text{ for } -l < x < l$$

Boundary Conditions:

$$T(-l,t) = T(l,t) = T_0 = 0 \text{ when } t > 0$$

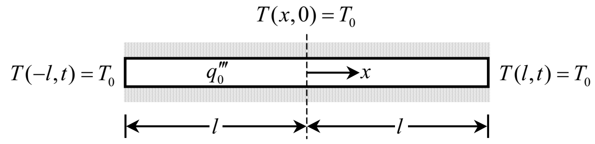


Figure 3. Physical situation of case I.

The analytical solution is given by [6]

$$\frac{T-T_0}{q_0'''l^2/k} = \frac{1}{2}(1-X^2) - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}e^{-\lambda_n^2 Fo}\cos(\lambda_n X) \tag{31}$$

where $Fo = \alpha\Delta t/l^2$ is Fourier number, $\alpha = k/\rho c_p$ is the thermal diffusivity, $\lambda_n = (2n+1)\pi/2$, and $X = x/l$.

The numerical and analytical solutions are plotted in Figure 4.

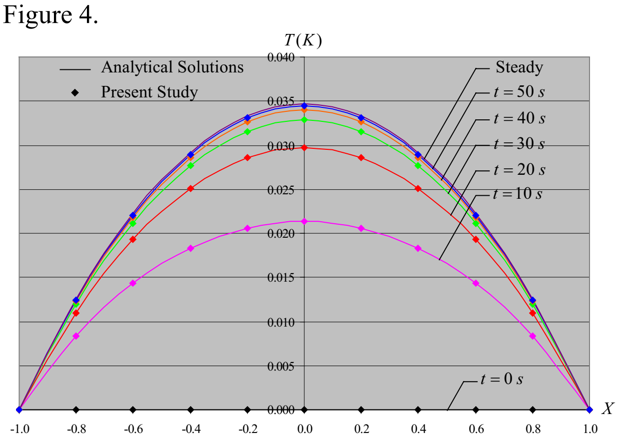


Figure 4. Numerical and analytical solutions of case I.

<u>Case II</u>

The physical situation of transient heat conduction with convection at both ends is shown in Figure 5.

Initial Condition:

$$T(x,0) = T_\infty = 0 \text{ for } -l < x < l$$

Boundary Conditions:

$$k(\partial T/\partial n) + hT = hT_\infty = 0 \text{ at } x = \pm l \text{ when } t > 0$$

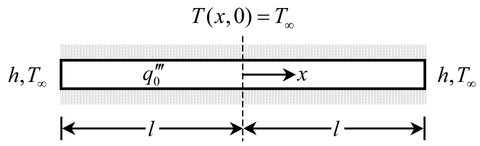


Figure 5. Physical situation of case II.

The analytical solution is given by [6]

$$\frac{T-T_\infty}{q_0'''l^2/k}Bi = 1 + \frac{Bi}{2} - \frac{Bi}{2}X^2$$

$$-2Bi^2\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2(\lambda_n^2+Bi^2+Bi)\cos\lambda_n}e^{-\lambda_n^2 Fo}\cos(\lambda_n X) \tag{32}$$

where $Bi = hl/k$ is Biot number, $\lambda_n \tan\lambda_n = Bi$ is the characteristic equation, and $X = x/l$.

The numerical and analytical solutions are plotted in Figure 6.

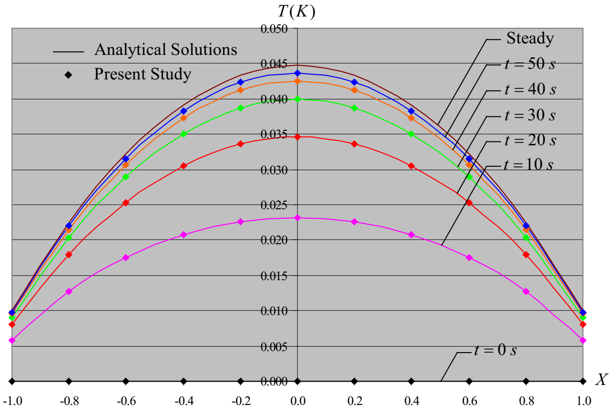


Figure 6. Numerical and analytical solutions of case II.

<u>Case III</u>

The physical situation of transient heat conduction with specified temperature at one end and convection at another end is shown in Figure 7.

Initial Condition:

$$T(x,0) = T_\infty = 0 \text{ for } 0 < x < l$$

Boundary Conditions:

$$T(0,t) = T_\infty = 0 \text{ at } x = 0 \text{ when } t > 0$$

$$k(\partial T/\partial n) + hT = hT_\infty = 0 \text{ at } x = l, \text{ when } t > 0$$

$$T(x,0) = T_\infty$$

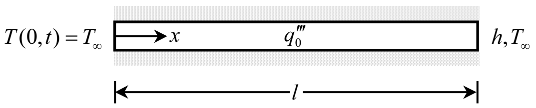


Figure 7. Physical situation of case III.

The analytical solution is given by [6]

$$\frac{T - T_\infty}{q_0''' l^2/k} = \frac{1 + Bi/2}{1 + Bi} X - \frac{X^2}{2} + 4Bi \sum_{n=1}^{\infty} \frac{1 - \cos\lambda_n}{\lambda_n^2(\lambda_n^2 + Bi^2 + Bi)\sin 2\lambda_n} e^{-\lambda_n^2 Fo} \sin(\lambda_n X) \qquad (33)$$

where $Bi = hl/k$ is Biot number, $\lambda_n \cot\lambda_n + Bi = 0$ is the characteristic equation, and $X = x/l$.

The numerical and analytical solutions are plotted in Figure 8.

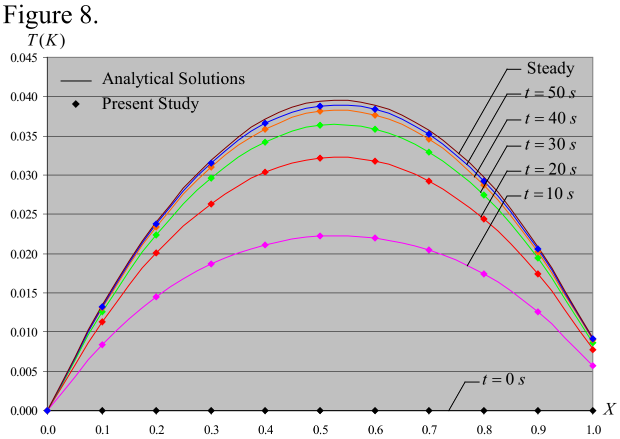


Figure 8. Numerical and analytical solutions of case III.

<u>Case IV</u>

The physical situation of transient heat conduction with one pulse source occurs at the beginning of time and specified temperatures at both ends is shown in Figure 9.

Initial Condition:

$$T(x,0) = T_0 = 0 \text{ for } 0 < x < l$$

Instantaneous pulse occurs at $x = x_1$ with strength $Q_0''$.

Boundary Conditions:

$$T(0,t) = T(l,t) = T_0 = 0 \text{ when } t > 0$$

$$T(x,0) = T_0$$

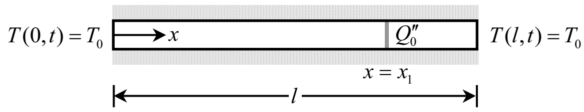


Figure 9. Physical situation of case IV.

The analytical solution is given by [6]

$$\frac{T - T_0}{Q_0 \alpha / kAl} = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 Fo} \sin(n\pi X_1)\sin(n\pi X) \qquad (34)$$

where $Fo = \alpha\Delta t/l^2$ is Fourier number, $\alpha = k/\rho c_p$ is the thermal diffusivity, $X_1 = x_1/l$, and $X = x/l$.

The numerical solutions just after the occurrence of the instantaneous pulse source are plotted in Figure 10 without the analytical solutions because they lead to oscillatory temperature distributions at small value of times. The numerical and analytical solutions after an elapsing time are plotted in Figure 11.

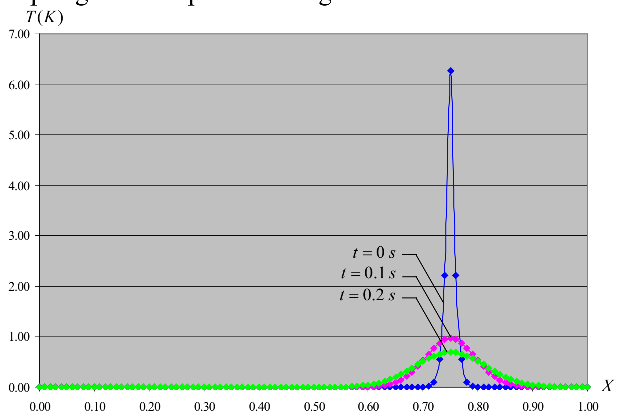


Figure 10. Numerical solutions of case IV just after the occurrence of the instantaneous pulse source.

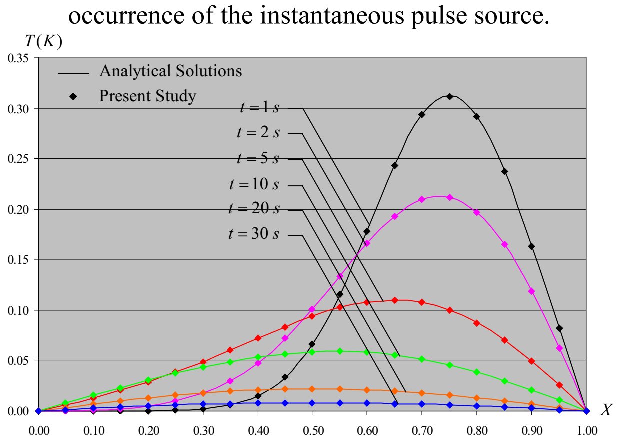


Figure 11. Numerical and analytical solutions of case IV after an elapsing time.

Case V

The physical situation of transient heat conduction with symmetrical pulse sources occurs at the beginning of time and convection at both ends is shown in Figure 12.

Initial Condition:

$$T(x,0) = T_\infty = 0 \text{ for } -l < x < l$$

Instantaneous pulses occur at $x = \pm x_1$ with strength $Q_0''$.

Boundary Conditions:

$$k(\partial T/\partial n) + hT = hT_\infty = 0 \text{ at } x = \pm l \text{ when } t > 0$$

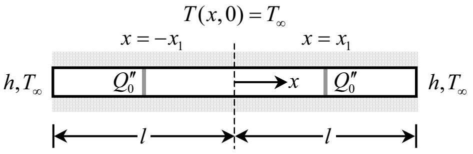


Figure 12. Physical situation of case V.

The analytical solution is given by [6]

$$\frac{T - T_\infty}{Q_0''\alpha/kl} = 2\sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_n + \sin\lambda_n \cos\lambda_n} \times e^{-\lambda_n^2 Fo}\cos(\lambda_n X_1)\cos(\lambda_n X) \quad (35)$$

where $\lambda_n \tan\lambda_n = Bi$ is the characteristic equation, $Bi = hl/k$ is Biot number, $X_1 = x_1/l$, and $X = x/l$.

Like in case IV, the numerical solutions just after the occurrence of the instantaneous pulse sources are plotted in Figure 13 and the numerical and analytical solutions after an elapsing time are plotted in Figure 14.

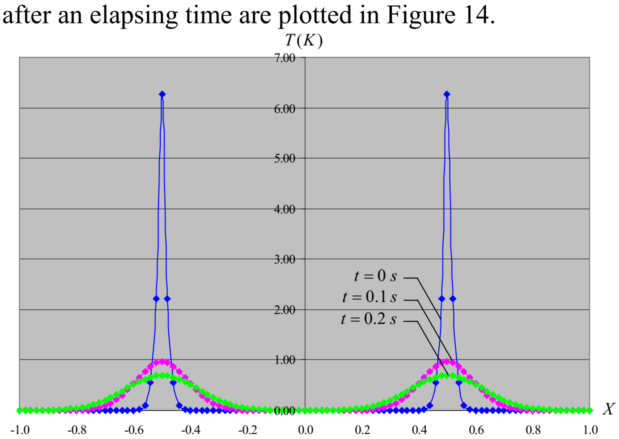


Figure 13. Numerical solutions of case V just after the occurrence of the instantaneous pulse source.

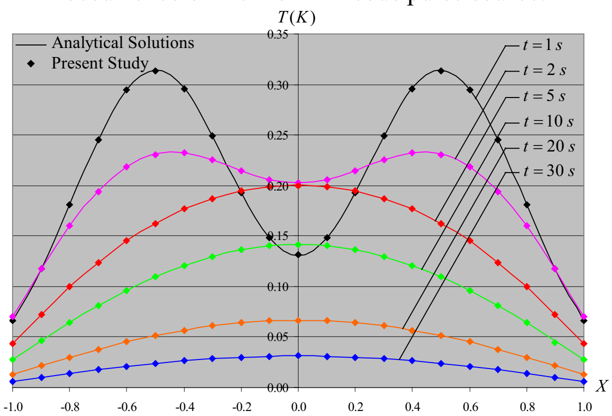


Figure 14. Numerical and analytical solutions of case V after an elapsing time.

Case VI

The physical situation of transient heat conduction with a continuous pulse source occurs at the beginning of time and specified temperatures at both ends is shown in Figure 15.

Initial Condition:

$$T(x,0) = T_0 = 0 \text{ for } 0 < x < l$$

Continuous pulse occurs at $x_1 \le x \le x_2$ with strength $Q_0'''$.

Boundary Conditions:

$$T(0,t) = T(l,t) = T_0 = 0 \text{ when } t > 0$$

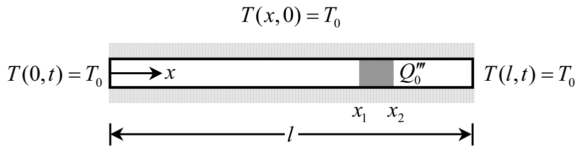


Figure 15. The physical situation of case VI.

The analytical solution is given by [6]

$$\frac{T - T_0}{Q_0'''\alpha/k} = 2\sum_{n=1}^{\infty} \frac{\cos(n\pi X_1) - \cos(n\pi X_2)}{n\pi} \times e^{-n^2\pi^2 Fo}\sin(n\pi X) \quad (36)$$

where $Fo = \alpha\Delta t/l^2$ is Fourier number, $\alpha = k/\rho c_p$ is the thermal diffusivity, $X_1 = x_1/l$, $X_2 = x_2/l$, and $X = x/l$.

The numerical solutions just after the occurrence of the continuous pulse source are plotted in Figure 16 and the numerical and analytical solutions after an elapsing time are plotted in Figure 17.

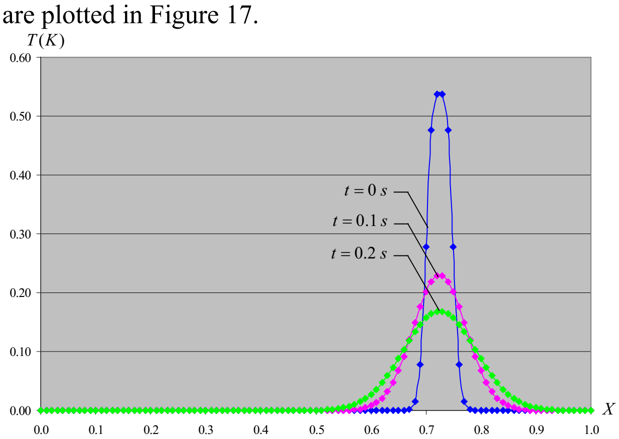


Figure 16. Numerical solutions of case VI just after the occurrence of the instantaneous pulse source.

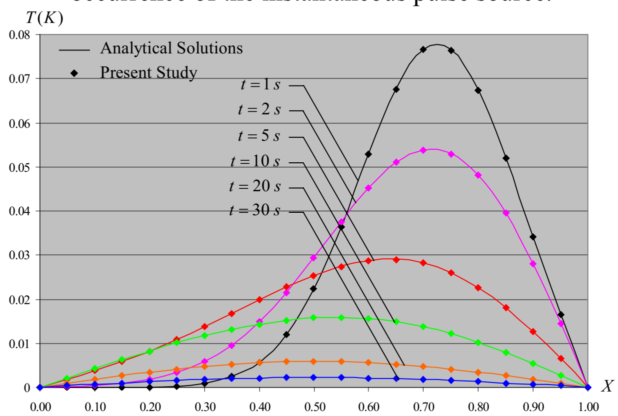


Figure 17. Numerical and analytical solutions of case VI after an elapsing time.

Six preceding example problems can be classified into two categories. The first three cases are the transient conduction heat transfer problems without instantaneous pulse sources within the spatial domain. The numerical results obtained from the developed computer program show good agreements with the analytical solutions both in local and global considerations. The remaining cases fall into another category, the transient conduction heat transfer problems with instantaneous pulse sources within the spatial domain. In solving problems of this type, the interelement nodes must coincide with all pulse sources and/or the edges of continuous pulse sources. A special treatment must be performed in the time coordinates just before the existence of pulse sources. Firstly, one or several pseudo-time steps are inserted into the time coordinates. Then, the heat fluxes at the interelement nodes for pulse sources and/or the rates of heat generation over the elements for continuous pulse sources are specified. The numerical value of specified heat fluxes and/or rates of heat generation can be found from the fact that the amount of generated energy in the pseudo-time interval must be equal to the strength of pulse sources and/or continuous pulse sources. After executing the developed computer program over the pseudo-time interval, the initial conditions are obtained and the remaining procedures are same as the procedures in solving the problems without instantaneous pulse sources. For the time coordinates just after the occurrence of pulse source, the analytical solutions lead to the oscillatory temperature distributions. Thus, the numerical and analytical solutions are compared only when the time coordinates are large enough. Although the numerical results are very closed to the analytical solutions after an elapsing time of the occurrence of the instantaneous pulse sources, the smaller pseudo-time interval for the special treatment is recommended in the time marching process.

## 5. Conclusion

A finite element analysis for transient one-dimensional conduction heat transfer problems has been presented. Space-time nine-node rectangular elements were employed to distinguish the concept of directly applied weighted-residual methods from other well-established standard methods. The corresponding finite element computer program than can be executed on standard personal computer was developed.

Six problems with various initial and boundary conditions, which can be categorized into two classes, were selected to verify the developed finite element computer program. For the problems without pulse sources within the spatial domain, the temperature distributions at each time step were directly obtained from the execution of the developed computer program. A special treatment was required when pulse sources exist within the spatial domain. The initial conditions can be generated by specifying heat fluxes and/or rates of heat generation that can generate the amount of energy equal to the pulse strength within the pseudo-time interval and executing the developed computer program over the pseudo-time interval. Then, the temperature distributions for the time after the occurrence of pulse sources can be obtained with the same procedures when solve the problems without pulse sources.

All numerical results showed good agreements with analytical solutions indicate that the accuracy of this method is excellent. Furthermore, the presented method can efficiently solve the problems that pulse sources exist in the spatial domain with the aid of a special treatment in the time marching process.

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