COLUMN BUCKLING BY A BOUNDARY INTEGRAL METHOD

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Abstract

Boundary integral methods are powerful numerical methods for numerous classes of engineering problems whose exact solutions cannot be obtained. The present paper presents the basic principle for a direct boundary integral method for one-dimensional eigenvalue problems with application to the classical column buckling. All the cases demonstrated herein, closed form critical loads are found.

Introduction

The boundary integral methods or boundary element methods have been employed for many years to solve various types of problems in science and engineering. Eventhough the formulations have the beauty of transforming the differential equations of the domain inside a body into equivalent sets of the integral ones of the bounding surface of that body, it is only recently that more attentiveness has been concentrated on these methods. This implies that any discritization scheme which might be necessary for a 3-dimensional problem would only be reduced to the subdivision of its bounding surface. As a result, several general numerical algorithms have been developed systematically with direct applications to wide range of practical and theoretical problems. Exhaustive references can be found in the text books such as references [1] to [3]. For many one-dimensional boundary value problems, exact closed form solutions by boundary integral methods can be obtained. This paper is entirely concerned with a demonstration of the principle and simplicity of the direct boundary integral method with special attention to the eigenvalue problem of classical column buckling. The types of columns considered herein are simply-supported, clamped-clamped, and clamped-free columns.

Formulation of the Problem

Consider a column of length L with uniform cross-section subject to end compression of magnitude P, the governing differential equation is

$$EIw_{,xxxx} + Pw_{,xx} = 0 \tag{1}$$

where comma indicates the differentiation with respect to the independent variable concerned. To convert the differential equation in the domain to an equivalent set of the integral equation, one can start by writing the following weighted residual statement which requires the vanishing of the inner product such that,

$$\int_o^L (EIw_{,xxxx} + Pw_{,xx})\, w^* \, dx = 0 \tag{2}$$

where $w^*(x)$ is an admissible weighting function and continuous through the order of the differentiation required by subsequent operations. Eq.(2) can be integrated by parts four times to yield,

$$\int_o^L \{EIw^*_{,xxxx} + Pw^*_{,xx}\} w dx + \{m\theta^* - Rw^* - m^*\theta + R^* w\}_o^L = 0 \tag{3}$$

where the slope, bending moment and shear force are,

$$\theta = w_{,x} \qquad m = -EIw_{,xx} \qquad R = -(Pw_{,x} + EIw_{,xxx})$$

and also analogous to the column problem one can define the following variables for the weighting function such that,

$$\theta^* = w^*_{,x} \qquad m^* = -EIw^*_{,xx} \qquad R^* = -(Pw^*_{,x} + EIw^*_{,xxx})$$

Equation (3) is considered to be the starting equation for the boundary integral mehtod. One can select the weighting function such that,

$$EI\, w^*_{,xxxx} + P w^*_{,xx} = 0 \tag{4}$$

Then eq.(3) becomes,

$$\left[\, m\theta^* - Rw^* - m^*\theta + R^* w \,\right]_o^L = 0 \tag{5}$$

Let $k^2 = P/EI$, the solution of eq.(4) is,

$$w^* = C_1 \sin kx + C_2 \cos kx + C_3\, x + C_4 \tag{6}$$

Substitute eq. (6) and its derivatives into eq. (5), one obtains the following algebraic equation,

$$\begin{aligned} 0 = {} & m(L)(C_1 k \cos kL + C_2\, k \sin kL + C_3) - m(0)(C_1 k + C_3) - \\ & R(L)(C_1 \sin kL + C_2 \cos kL + C_3 L + C_4) + R(0)(C_2 + C_4) - \\ & \theta(L) k^2 EI(C_1 \sin kL + C_2 \cos kL) + \theta(0)(k^2 EIC_2) + \\ & w(L)\{-P(C_1 k \cos kL - C_2 k \sin kL + C_3) + k^3 EI(C_1 \cos kL - C_2 \sin kL)\} + \\ & w(0)\{P(C_1 k + C_3) - EIk^3 C_1\} \end{aligned} \tag{7}$$

Once the boundary conditions have been specified, eq. (7) will lead to the solution of the column buckling problem.

Solutions

1. Simply-Supported Column

For a column with two ends being simply-supported, the boundary conditions are,

$$w(0) = w(L) = m(0) = m(L) = 0$$

Substitute the above boundary conditions into eq.(7), since C_i are independent, therefore the following four equations are obtained,

$$\begin{bmatrix} 0 & -k^2EI\sin kL & 0 & -\sin kL \\ k^2EI & -k^2EI\cos kL & 1 & -\cos kL \\ 0 & 0 & 0 & -L \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta(0) \\ \dot{\theta}(L) \\ R(0) \\ R(L) \end{bmatrix} = 0$$

For a non-trivial solution, the determinant of the above equation must be zero, from which one gets,

$$\sin kL = 0$$

or

$$kL = n\pi \quad , \quad n = 1,2,3,\ldots$$

or

$$P = \left(\frac{n\pi}{L}\right)^2 EI$$

or

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

2. Clamped Column

For a column with two clamped ends, the boundary conditions are,

$$w(0) = w(L) = \theta(0) = \theta(L) = 0$$

Substitute into eq.(7) yields the following four equations,

$$\begin{bmatrix} -k & k\cos kL & 0 & -\sin kL \\ 0 & -k\sin kL & 1 & -\cos kL \\ -1 & 1 & 0 & -L \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} m(0) \\ m(L) \\ R(0) \\ R(L) \end{bmatrix} = 0$$

For a non-trivial solution, the determinant of the above equation must vanish. Expanding the determinant and after some simplification one has,

$$\sin\frac{kL}{2}\left(-4k\sin\frac{kL}{2} + 2k^2L\cos\frac{kL}{2}\right) = 0$$

Hence, either

$$\sin kL = 0 \qquad (8)$$

or

$$-4k\sin\frac{kL}{2} + 2k^2L\cos\frac{kL}{2} = 0 \qquad (9)$$

Eq.(8) yields,

$$\frac{kL}{2} = n\pi\ , \qquad n = 1,2,3,\ldots$$

or

$$P = \frac{4n^2\pi^2EI}{L^2}$$

or

$$P_{cr} = \frac{4\pi^2EI}{L^2}$$

However, eq.(9) leads to $\tan\frac{kL}{2} = \frac{kL}{2}$ which gives $P_{cr} > \frac{4\pi^2EI}{L^2}$

3. Clamped-Free Column

The boundary conditions for a clamped-free column are,

$$w(0) = \theta(0) = m(L) = R(L) = 0$$

Hence, eq.(7) becomes,

$$\begin{bmatrix} 0 & EIk^2 \sin kL & k & 0 \\ 0 & -EIk^2 \cos kL & 0 & 1 \\ P & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w(L) \\ \theta(L) \\ m(0) \\ R(0) \end{bmatrix} = 0$$

For a non-trivial solution yields,

$$\cos kL = 0$$

or

$$kL = \left(\frac{2n-1}{2}\right)\pi \quad , \quad n = 1,2,3,\ldots$$

or

$$P = \left(\frac{2n-1}{2}\right)^2 \frac{\pi^2 EI}{L^2}$$

or

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$

Conclusion

The boundary integral method is another powerful method for determining the solution of eigenvalue problems. The paper presents the basic principle with an application to the classical column buckling. For all cases demonstrated, exact critical compressive loads are obtained.

References

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3. Michihiro Kitahara, Boundary Integral Equation Methods in Eigenvalue Problems of Elastodynamics and Thin Plates, Elsevier, 1985.