Application of Space-Time Finite Element Method for Two-Dimensional Heat Conduction Problems

Passakorn Vessakosol

Department of Mechanical Engineering, Prince of Songkla University,
Hatyai, Songkla, Thailand, 90110
E-mail: passakornv@hotmail.com, Tel. 074 287035-6, Fax. 074 558830

Abstract

A space-time finite element method is presented for solving transient two-dimensional heat conduction problems. The finite element formulation in this paper corresponds to the Galerkin weighted residual method. The linear hexahedron is employed for interpolating the unknown quantities. A computer program is developed to verify the concept by comparing the current numerical result with the analytical result.

***Keywords**:* finite element, transient, conduction.

1. Introduction

The time stepping method is commonly combined with the finite element method for solving the parabolic partial differential equation which associated with time-dependent problems. Normally, the temporal approximation is performed by the finite difference method and the spatial approximation is accomplished by the finite element method. Although this method is easy to implement, it can be expensive if steep gradients occur in the solution. Additionally, the stability of the solution must be controlled as well as the problem of controlling the global error [2].

In this paper, the space-time finite element method is proposed to solve the time-dependent temperature distribution in two-dimensional body. The method of weighted-residual is employed with elements both in space and time coordinates [1]. This method characterized as the implicit time stepping method which the numerical result is unconditionally stable.

2. Space-time element

The heat conduction considered in this paper depends on both space co-ordinates and time. The space-time domain is proposed as the product of spatial interval and time interval. This domain is divided into several time slabs as illustrated in Fig. 1. The definition of time interval I is given by

$$I = \{t : 0 < t < T\} \tag{1}$$

where T is the given time for numerical calculation and the spatial interval R can be described as follows

$$R = \{x : 0 < x < A,\ y : 0 < y < B\} \tag{2}$$

where A and B are the maximum sizes of material in x and y directions, respectively. For the nth space-time domain, the spatial domain is divided into n_e elements, R_n^e, $e =$ 1,2,3,… , n_e. Therefore, the nth space-time domain E_n with boundary Γ_n are defined as

$$E_n = R \times I_n \tag{3}$$

$$\Gamma_n = S \times I_n \tag{4}$$

where R is the two-dimensional spatial domain with boundary S; I_n is the time interval between t_n and t_{n+1}.

3. Numerical formulation

3.1 Heat conduction equation

The two-dimensional transient heat conduction in a solid body is governed by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + Q = \rho c\frac{\partial T}{\partial t} \tag{5}$$

where $T(x, y, t)$ is the unknown temperature that varies as the function of the spatial coordinates, x and y , and time, t ; k is the thermal conductivity of material; ρ is the density of material; c is the specific heat of material; Q is the volumetric heat generation rate. Other important parameter need to be defined here is the thermal diffusivity, $\alpha = k/\rho c$. To complete the problem, the boundary and initial conditions are required. These conditions are given in the form

$$T = \bar{T} \quad \text{on } S_1 \tag{6a}$$

$$-k\left(\frac{\partial T}{\partial x} l_x + \frac{\partial T}{\partial y} l_y\right) = \bar{q} \quad \text{on } S_2 \tag{6b}$$

$$-k\left(\frac{\partial T}{\partial x} l_x + \frac{\partial T}{\partial y} l_y\right) = h\left(T - T_\infty\right) \quad \text{on } S_3 \tag{6c}$$

$$T(x, y, 0) = T_0(x, y) \tag{6d}$$

where $S = S_1 + S_2 + S_3$; S_1 is the part of boundary on which $\bar{T}$, temperature, is specified; S_2 is the part of boundary on which $\bar{q}$, the flux of heat, is specified; S_3 is the part of boundary on which $h(T - T_\infty)$, the convection heat transfer, is specified; h is the heat transfer coefficient; T_∞ is the temperature of ambient fluid; l_x and l_y are the direction-cosine components in x and y directions, respectively, of outward normal vector to the boundary.

3.2 Finite element formulation

The weak form of the governing equation is developed by using a weighted residual approach called the standard Galerkin method. Eq. (5) is multiplied by N_i^e, the interpolation function of element, and then the resulting equation is integrated over the element domain E^e:

$$\int_{E^e} N_i^e \left\{ \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + Q - \rho c \frac{\partial T}{\partial t} \right\} dE^e = 0 \tag{7}$$

After applying the divergence theorem and integration by parts, Eq. (7) becomes

$$\int_{E^e}\left(-\rho c N_i^e \frac{\partial T^e}{\partial t} - k \frac{\partial T^e}{\partial x}\frac{\partial N_i^e}{\partial x} - k \frac{\partial T^e}{\partial y}\frac{\partial N_i^e}{\partial y} \right) dE^e + \int_{S_2} \bar{q} N_i^e dS_2^e + \int_{S_3} N_i^e h\left(T^e - T_\infty\right) dS_3^e + \int_{E^e} Q N_i^e dE^e = 0 \quad , i = 1,2,\ldots,8 \tag{8}$$

The temperature value and its gradients at any point of space-time element in Fig. 2 is interpolated by

$$T^e(x, y, t) = \sum_{k=1}^{8} N_k^e(x, y, t) T_k \,, \tag{9a}$$

$$\frac{\partial T^e}{\partial x} = \sum_{k=1}^{8} \frac{\partial N_k^e}{\partial x} T_k \,, \quad \frac{\partial T^e}{\partial y} = \sum_{k=1}^{8} \frac{\partial N_k^e}{\partial y} T_k \tag{9b}$$

Substituting Eqs. (9a) – (9b) into Eq. (8) yields

$$[D]\{T\} = \{f\} \tag{10}$$

where

$$[D] = \int_{E^e} k \left(\frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} \right) dE^e + \int_{E^e} \rho c_p N_i^e \frac{\partial N_j^e}{\partial t} dE^e - \int_{S_3} h N_i^e N_j^e dS_3^e \tag{11}$$

$$\{f\} = \int_{S_2} \bar{q} N_i^e dS_2^e - \int_{S_3} N_i^e h T_\infty dS_3^e + \int_{E^e} Q N_i^e dE^e \tag{12}$$

$$\{T\} = \{T_{i,j,n}\ T_{i+1,j,n}\ T_{i+1,j+1,n}\ T_{i,j+1,n}\ T_{i,j,n+1}\ T_{i+1,j,n+1}\ T_{i+1,j+1,n+1}\ T_{i,j+1,n+1}\} \tag{13}$$

The integrals in the element matrix and element vector of Eq. (10) are numerically evaluated by means of the Gauss-Legendre technique with 2 x 2 x 2 points in natural coordinate system with the help of coordinate transformation [2] through the jacobian matrix. The coordinates of the *i*th nodal points of the master hexagonal element are given in Table 1. In this paper, the element matrices and vectors are evaluated implicitly and easily by the numerical integration procedure. There is no need to derive the element matrices and vectors of hexahedron element explicitly. The global matrix of variables at *n*th and *n*-1th time step is obtained by assembling all element matrices. Since the values of temperature are known at t_0, values of temperature at $t_1 = t_0 + \Delta t$ can be obtained after applying the boundary and initial conditions and then solving the system of simultaneous linear algebraic equations. At each new time step, an identical calculation procedure will be used until a required time is reached.

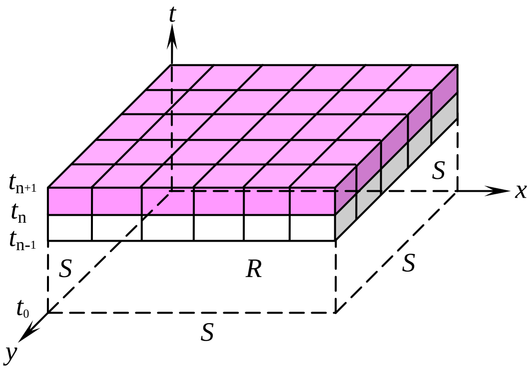

Fig. 1 Space-time discretization

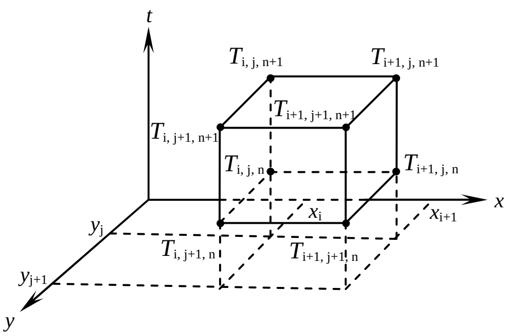

Fig. 2 A hexahedron element

Table 1 Coordinates of (ξ_i, η_i, ζ_i)

Node	ξ_i	η_i	ζ_i
1	–1	–1	–1
2	1	–1	–1
3	1	1	–1
4	–1	1	–1
5	–1	–1	1
6	1	–1	1
7	1	1	1
8	–1	1	1

4. Numerical results

Four test cases whose solution geometry is square to verify the accuracy of the developed computer program based on the space-time finite element concept. The volumetric heat generation rate of this study is equal to zero. The finite element model of 100 hexahedron elements and 242 nodes, 121 nodes at the current time n and 121 nodes at the next time n+1, with 11 nodes equally spaced in each x and y directions. The first case is related to a one-dimensional transient analysis in semi-infinite medium and other three cases are focused on two-dimensional transient analysis in a square plate with different boundary and initial conditions. The numerical results are compared with those of analytical solutions.

For case 1, a constant heat flux $q_0'' = 10^5$ W/m^2 is imposed on the left boundary surface at initial time and maintained for $t > 0$. Other surfaces of the plate are insulated except the right surface which its temperature is kept to 35°C. Also, the initial temperature of domain is 35°C. The length of square plate is 0.1 m. The boundary and initial conditions are shown in Fig. 3(a). The analytical solution for this case [5] is

$$T(x,t) = T_0 + \frac{2q_0''\sqrt{\alpha t/\pi}}{k}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{q_0'' x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{14}$$

where erfc(x) is the complementary error function; q_0'' is the constant heat flux. The analytical and numerical solutions are plotted in Fig. 4.

For case 2, Fig. 3(b) shows the geometry, parameters and conditions of the square plate. All the boundary surfaces are kept at zero temperature. The length of square plate is 1 m. The analytical solution for the second case [2] is

$$T(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A_{ij}\sin(\lambda_i x)\sin(\beta_j y)\cdot \exp\left(-\alpha\left(\lambda_i^2 + \beta_j^2\right)t\right) \tag{15}$$

where

$$\lambda_i = \frac{i\pi}{W},\quad \beta_j = \frac{j\pi}{W},$$
$$A_{ij} = \frac{4T_0}{ij\pi^2}\left((-1)^{i+1}-1\right)\left((-1)^{j+1}-1\right)\cdot \tag{16}$$

The comparison between exact and numerical solutions is shown in Fig. 5. Fig. 5(b) shows the stable numerical solution of space-time finite element method which violates the maximum size of time step in explicit scheme. In two-dimensional heat conduction problems, the limit of time step size for explicit numerical scheme [6] is $0.25\rho c_p(\Delta x)^2/k$ where Δx = Δy to produce the meaningful solutions. The maximum time step size for case 2 is about 30 sec.

For case 3, the top and right surface temperatures are maintained at 1°C while other boundaries are insulated as shown in Fig. 3(c). The length of square plate is 1 m. The analytical solution for the third case [1] is

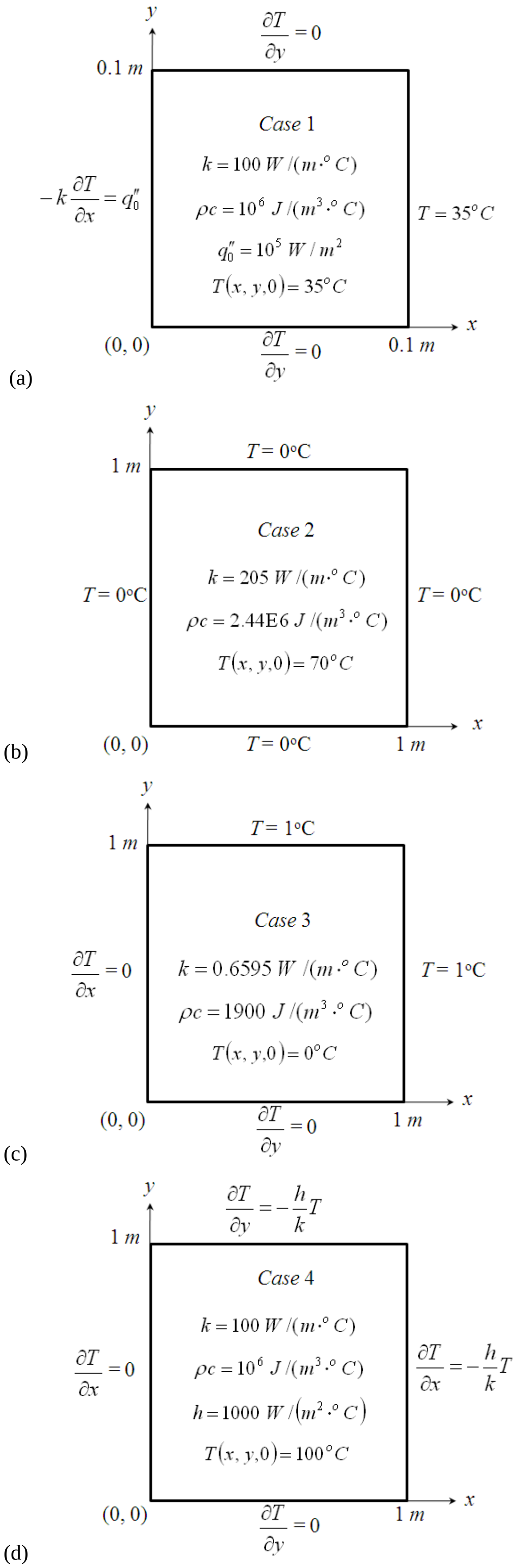

Fig. 3 (a) case 1 (b) case 2 (c) case 3 (d) case 4

$$T(x,y,t)=1+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}B_{ij}\cos(\lambda_i x)\cos(\beta_j y)\cdot \exp\left(-\alpha\left(\lambda_i^2+\beta_j^2\right)t\right) \quad (17)$$

where

$$\lambda_i=\frac{(2i-1)\pi}{2W},\quad \beta_j=\frac{(2j-1)\pi}{2W},$$
$$B_{ij}=\frac{-16(-1)^{i+1}(-1)^{j+1}}{\pi^2(2i-1)(2j-1)}. \quad (18)$$

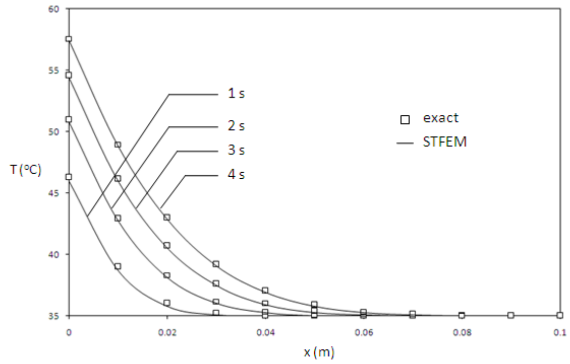

Fig. 4 Case 1: Comparison of space-time finite element and exact solutions at the points of the middle of the plate ($y=0.05\,m$) with $\Delta t=0.1$sec.

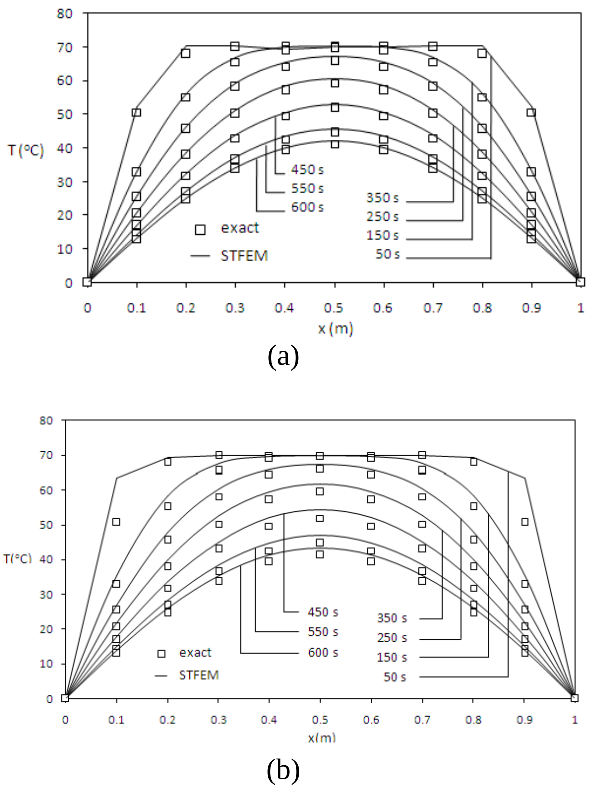

Fig. 5 Case 2: Comparison of space-time finite element and exact solutions at the points of $x=0.5\,m$ with (a) Δt = 10 sec and (b) Δt = 50sec.

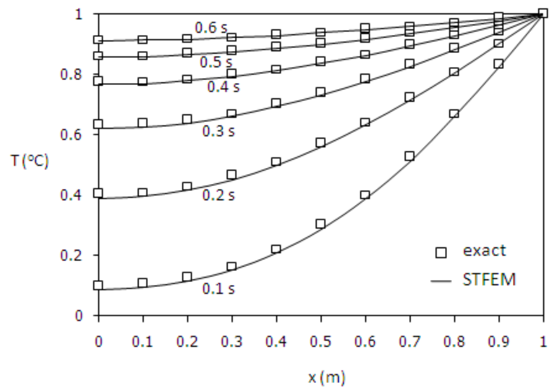

Fig. 6 Case 3: Comparison of space-time finite element and exact solutions at the points of left boundary ($x = 0$) with $\Delta t = 0.01\,\text{sec}$.

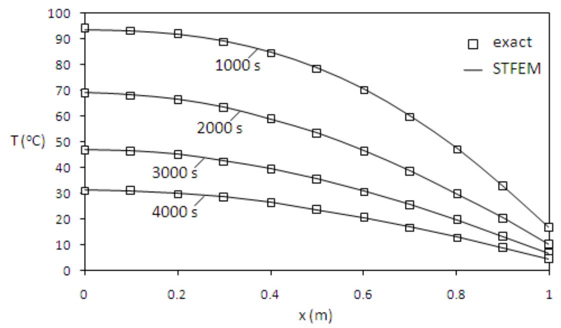

Fig. 7 Case 4: Comparison of space-time finite element and exact solutions at the points of left boundary ($x = 0$) with $\Delta t = 100\,\text{sec}$.

Fig. 6 gives the transient temperature distribution obtained from the analytical and numerical methods.

For case 4, the top and right surfaces are subjected to the convective heat transfer whereas the other surfaces are insulated as shown in Fig. 3(d). The length of square plate is 1 m. The analytical solution for the fourth problem [4] is

$$T(x,y,t) = 4T_0 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\bar{H}^2 \exp\left(-\alpha\left(\lambda_i^2+\beta_j^2\right)t\right)}{\left(W\left(\lambda_i^2+\bar{H}^2\right)+\bar{H}\right)\left(W\left(\beta_j^2+\bar{H}^2\right)+\bar{H}\right)} \cdot \left(\frac{\cos(\lambda_i x)\cos(\beta_j y)}{\cos(\lambda_i W)\cos(\beta_j W)}\right) \quad (19)$$

where

$$\lambda_i \tan(\lambda_i W) = \frac{h}{k},\ \beta_j \tan(\beta_j W) = \frac{h}{k},\ \bar{H} = \frac{h}{k} \quad (20)$$

Fig. 7 shows a comparison of the space-time finite element solution with the analytical solution. As shown in Figs. 4 - 7, the space-time finite element solutions agree well with analytical solutions in all test cases.

5. Conclusions

A space-time finite element method based on Galerkin weighted residual approach is presented to predict transient temperature field of two-dimensional heat conduction model. The proposed method is verified with four examples of different model parameters, boundary conditions and initial conditions. The results from analytical method and space-time finite element method are compared. The numerical results for all problem cases agree well with their corresponding analytical results. The current approach has only been tested for linear problems. In the future, the space-time finite element method should be modified to solve nonlinear transient heat conduction problems or the transient convection-diffusion problems.

6. References

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