Parabolic Backstepping Boundary Control: An Application to One-Dimensional Heat-generating Rod Temperature control

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Abstract

This paper applies a backstepping boundary control technique to stabilize the temperature of a slender rod. The rod is modeled by a parabolic partial differential equation (PDE) with Neumann conditions. The rod also includes an internal heat generator that makes the system unstable. For a feedback purpose, a Luenberger-like observer is used to estimate the temperatures along the rod. A gain kernel of the system is calculated and then used in the control law. In the experimental study, a copper rod with heat-generation is used as the control plant. The control set-up is anti-collocation. The temperature signal at one end is sent to the observer for estimation of the temperatures along the rod. The estimated values are then fed to the controller. Results from simulations using a finite difference method and from the real plant are compared and used to confirm the effectiveness of the control method.

***Keywords**:* Parabolic PDE, Backstepping, Boundary control

1. Introduction

Actual mechanical systems are governed by a partial differential equation. To control the systems, an ordinary differential approximation is usually utilized to obtain a control law. [1-3]

Recently, research on the infinite-dimensional control has been investigated. In [4-7], the infinite-dimensional backstepping boundary control for parabolic PDE systems has been introduced. However, its implementations are still very few.

This paper presents an experiment study of the infinite-dimensional backstepping boundary control scheme. It is conducted on a slender copper rod with a heat generator inside. The system is modeled by the parabolic PDE with Neumann conditions. The control set-up is anti-collocation (i.e., the sensor is at one end of rod and the actuator is located at the opposite end). A thermo-electric cooler (TEC) is used as the actuator.

The control objective is to stabilize the arbitrary temperature distribution along the rod to about the equilibrium temperature of the plant. Only temperature at one end is measured.

2. Mathematical Model

We consider the problem of heat conduction in a small cross-section copper rod with an internal heat generator. The rod has the length of L and the constant cross-section of A. The heat transfer from the rod is dissipated to its surrounding by convection. [4]

The resistance of the heat generator is linearly increased with temperature as [4,9]

$$R_e = R_1\,[1+\alpha_e(T-T_1)] \tag{1}$$

where

R_e	is the electric resistivity
R_1	is the electric resistivity at T_1.
T_1	is the temperature around which the R_e is linearized.
α_e	is the thermal coefficient of electric resistivity

From the law of conservation of energy and the Fourier's law, we get the heat equation [1,9]

$$\rho c_p T_t A d\ell = K_0 T_{\ell\ell}\, A d\ell - H\, p(T-T_0)\, d\ell + \frac{I^2 R_1}{A}[1+\alpha_e(T-T_1)]\, d\ell \tag{2}$$

where

T	is the rod temperature
A	is the rod cross-sectional area
p	is the perimeter of the rod
T_0	is the surrounding temperature
K_0	is the conductivity
ℓ	is the spatial variable
t	is the temporal variable
H	is the surface conductivity
ρ	is the rod mass density.
c_p	is the specific heat.

The subscripts t and ℓ mean partial differentiation with respect to time and space, respectively.

Dividing (2) with $\rho c_p A d\ell$, the equation becomes

$$T_t(\ell,t) = kT_{\ell\ell}(\ell,t) - \nu(T(\ell,t) - T_0) + B[1-\alpha_e(T(\ell,t)-T_1)] \tag{3}$$

where $k = \dfrac{K_0}{\rho c_p}$, $\nu = \dfrac{Hp}{\rho c_p A}$ and $B = \dfrac{I^2}{\rho c_p A^2} R_1$

We define the dimensionless variables of length, time and temperature, respectively, [4]

$$x = \frac{\ell}{L}, \quad \tau = \frac{t}{L^2/k}, \quad u = \frac{T - T_0}{T_e}$$

where $x \in [0,1]$ and

$$T_e = T_0 + B\frac{\alpha_e(T_1 - T_0) - 1}{B\alpha_e - \nu}$$

T_e is the constant equilibrium temperature distribution along the rod.
The dimensionless form of (3) is as follows

$$u_\tau(x,\tau) = u_{xx}(x,\tau) + \lambda u(x,\tau) \tag{4}$$

where

$$\lambda = L^2 \frac{(B\alpha_e - \nu)}{k}$$

3. Boundary control

Consider the parabolic heat equation with Neumann conditions [5,8]

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{5}$$

$$u_x(0,t) = 0 \tag{6}$$

$$u(1,t) = U(t) \tag{7}$$

The equation (5) is similar to (4) but the variable τ is replaced by t, for convenience.
The plant (5)-(7) with u(1,t) = 0 is unstable for a sufficiently large value of λ in the reaction term of the equation. The variable U(t) is the input to be designed. The unstable plant is shown in Fig. 1.

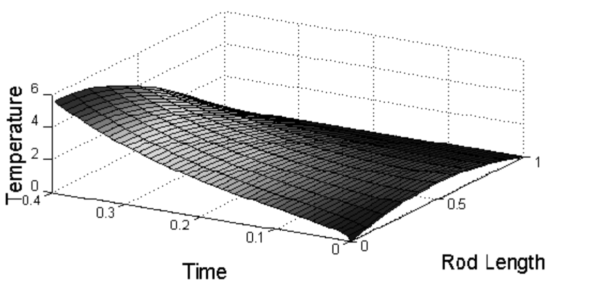

Fig. 1 The unstable plant.

In backstepping method, we use the following coordinate transformation

$$v(x,t) = u(x,t) - \int_0^x k(x,\xi)u(\xi,t)\,d\xi \tag{8}$$

To transforms the system (5)-(6) into the target system

$$v_t(x,t) = v_{xx}(x,t) \tag{9}$$

$$v_x(0,t) = 0 \tag{10}$$

$$v(1,t) = 0 \tag{11}$$

which is known to be exponentially stable. We can find the gain kernel $k(x,\xi)$ by substituting (8) into (9) and with the help of the Leibniz integral rule and the integration by-part, we get the following hyperbolic PDE:

$$k_{xx}(x,\xi) - k_{\xi\xi}(x,\xi) = \lambda k(x,\xi) \tag{12}$$

$$k_x(x,0) = 0 \tag{13}$$

$$k(x,x) = -\frac{\lambda}{2}x \tag{14}$$

where $(x,\xi) \in T = \{x,\xi : 0 < \xi < x < 1\}$

By transforming (12)-(13) into an integral equation and using the method of successive approximation, the solution of this PDE is [1]

$$k(x,\xi) = -\lambda \frac{I_1\left(\sqrt{\lambda(x^2-\xi^2)}\right)}{\sqrt{\lambda(x^2-\xi^2)}} \tag{15}$$

where I_1 is a first-order modified Bessel function.
In Fig. 2, the gain kernel $k(x,\xi)$ is plotted for several values of λ.
From the transformation (8) and the boundary condition (11) give the controller in the form

$$u(1,t) = -\int_0^1 k(1,\xi)u(\xi,t)\,d\xi$$

$$u(1,t) = -\int_0^1 \lambda \frac{I_1\left(\sqrt{\lambda(1-\xi^2)}\right)}{\sqrt{\lambda(1-\xi^2)}} u(\xi,t)\,d\xi \tag{16}$$

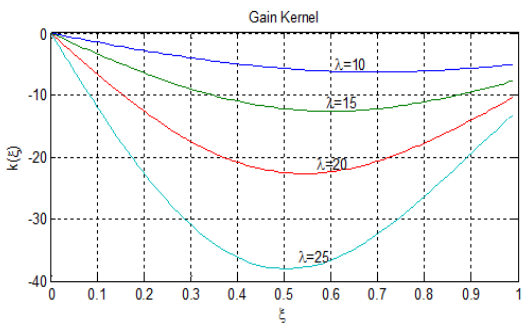

Fig. 2 Gain kernel with several values of λ.

3. Observer Design

The backstepping method in the last section requires complete measurements of the temperatures inside the domain for feedback. However, the only possible measurement of the system is at x = 0. The following observer is used to estimate the temperature along the rod, [6-8]

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \lambda \hat{u}(x,t) + q_1(x)[u(0,t) - \hat{u}(0,t)] \quad (17)$$

$$\hat{u}_x(0,t) = q_{10}[u(0,t) - \hat{u}(0,t)] \quad (18)$$

$$\hat{u}(1,t) = u(1,t) \quad (19)$$

Let the observer error be $\tilde{u} = u - \hat{u}$, we get the following PDE:

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \lambda \tilde{u}(x,t) + q_1(x)\tilde{u}(0,t) \quad (20)$$

$$\tilde{u}_x(0,t) = -q_{10}\tilde{u}(0,t) \quad (21)$$

$$\tilde{u}(1,t) = 0 \quad (22)$$

We use the coordinate transformation

$$\tilde{u}(x,t) = \tilde{v}(x,t) - \int_0^x q(x,\xi)\tilde{v}(\xi,t)\,d\xi \quad (23)$$

to transform system to

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) \quad (24)$$

$$\tilde{v}_x(0,t) = 0 \quad (25)$$

$$\tilde{v}(1,t) = 0 \quad (26)$$

Substitute (23) into (20)-(22) to get the kernel $q(x,\xi)$

$$q_{\xi\xi}(x,\xi) - q_{xx}(x,\xi) = \lambda q(x,\xi) \quad (27)$$

$$\frac{d}{dx}q(x,x) = \frac{\lambda}{2} \quad (28)$$

$$q(1,\xi) = 0 \quad (29)$$

which lead to

$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) - q_1(x,0)\tilde{v}_x(0,t) + [q_\xi(x,0) - q_1(x)]\tilde{v}(0,t) \quad (30)$$

$$\tilde{v}_x(0,t) = [q(0,0) - q_{10}]\tilde{v}(0,t) \quad (31)$$

$$\tilde{v}(1,t) = 0 \quad (32)$$

From (30)-(32) and (24)-(26), we get the observer gains

$$q_1(x) = q_\xi(x,0), \quad q_{10} = q(0,0) \quad (33)$$

Introduce the new variables [6-8]

$$\bar{x} = 1 - \xi, \quad \bar{\xi} = 1 - x, \quad \bar{q}(\bar{x},\bar{\xi}) = q(x,\xi)$$

The PDE (27)-(29) become

$$\bar{q}_{\bar{\xi}\bar{\xi}}(\bar{x},\bar{\xi}) - \bar{q}_{\bar{x}\bar{x}}(\bar{x},\bar{\xi}) = \lambda \bar{q}(\bar{x},\bar{\xi}) \quad (34)$$

$$\bar{q}(\bar{x},0) = 0 \quad (35)$$

$$\bar{q}(\bar{x},\bar{x}) = -\frac{\lambda}{2}\bar{x} \quad (36)$$

This PDE has the solution

$$\bar{q}(\bar{x},\bar{\xi}) = -\lambda \bar{\xi}\,\frac{I_1\left(\sqrt{\lambda(\bar{x}^2 - \bar{\xi}^2)}\right)}{\sqrt{\lambda(\bar{x}^2 - \bar{\xi}^2)}} \quad (37)$$

and is expressed in the original variables,

$$q(x,\xi) = -\lambda(1-x)\frac{I_1\left(\sqrt{\lambda(2-x-\xi)(x-\xi)}\right)}{\sqrt{\lambda(2-x-\xi)(x-\xi)}} \quad (38)$$

From (33), we get the observer gain function,

$$q_1(x) = \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \quad (39)$$

and observer gain constant,

$$q_{10} = -\frac{\lambda}{2} \quad (40)$$

Observer gain (39) and (40) will be used in observer (17)-(19)

4. Experimental set-up

A small copper tube with outside diameter of 5 mm. and 80 mm. long is used as the control plant. Seven temperature sensors are used to measure the temperatures along the rod. Note that only one at the rod's end (sensor No.1) will be used for feedback. It is installed with thermal insulation. Another five sensors are placed along the rod and their temperatures used for display only. At the other end of the rod, a thermo-electric cooler (TEC) based on a Peltier effect is installed and the last sensor (sensor No. 7) is attached for measuring the control temperature. NI-USB 6225 is used as data acquisition and control unit. The control diagram and the photo of experimental set-up are shown in Figs 3 and 4, respectively.

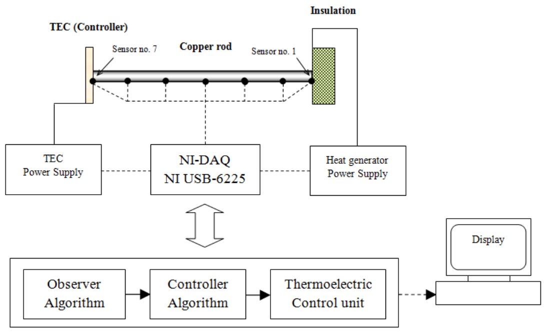

Fig. 3 Control diagram.

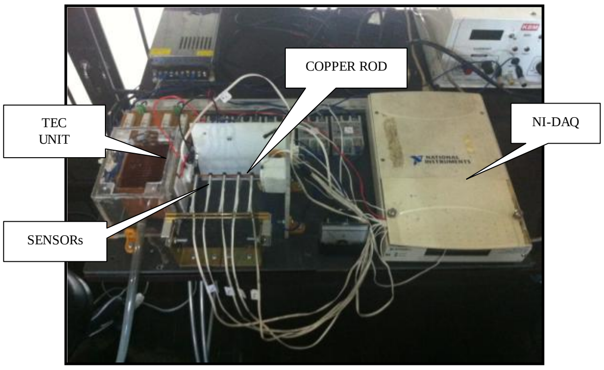

Fig. 4 Experimental set-up.

5. Results

The value of $\lambda = 12$ is set to make the system unstable. The heat generator in the rod is turned on about 10 minutes to create the initial temperature profile on the rod. The control algorithm is then started to control the temperatures to the equilibrium temperature.

Figs. 5, 6 and 7 show the control temperature at sensors No.7, 1 and 5, respectively. The graph displays the comparison between real rod temperatures, simulation and observation.

All cases yield similar results. The stability was obtained. Fig. 8 shows Temperature distribution along the rod before, during and at the end of control.

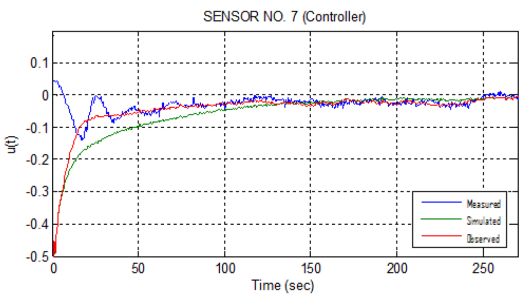

Fig. 5 Temperature at sensor No. 7 shows the control action.

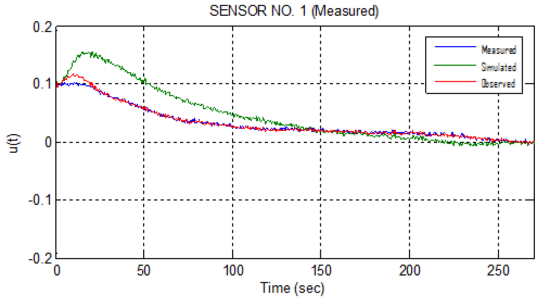

Fig. 6 Temperature at sensor No. 1.

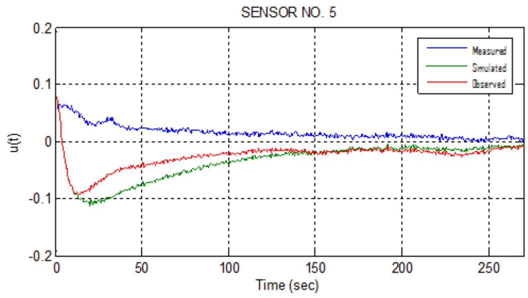

Fig. 7 Temperature at sensor No. 5.

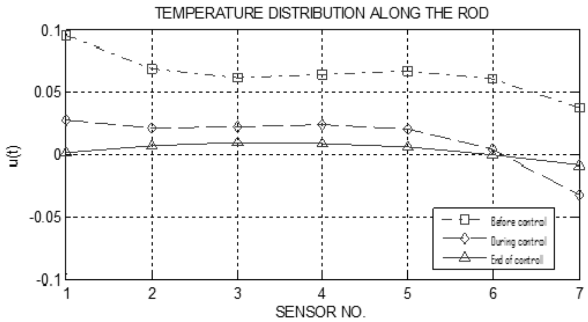

Fig. 8 Temperature distribution along the rod. Before, during and at the end of control.

6. Conclusions

An application of a backstepping boundary control technique to stabilize the temperature of a slender rod has been presented. Simulation and experimental results illustrate that the control method is effective.

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